Applications of Minor Summation Formula III, Plücker Relations, Lattice Paths and Pfaffian Identities

Masao ISHIKAWA and Masato WAKAYAMA

*Department of Mathematics, Faculty of Education, Tottori University

†Faculty of Mathematics, Kyushu Unversity

Abstract

The initial purpose of the present paper is to provide a combinatorial proof of the minor summation formula of Pfaffians in [8] based on the lattice path method. The second aim is to study applications of the minor summation formula for obtaining several identities. Especially, a simple proof of Kawanaka's formula concerning a q-series identity involving Schur functions [15] and of the identity in [16] which is regarded as a determinant version of the previous one are given.
2000 Mathematics Subject Classification : Primary 05E05, Secondly 05E10, 05Exx, 05A30,
Key Words : Pfaffian, Plücker relations, Lewis Carroll formula, partition, lattice paths method, minor summation formula, Cauchy identity, Schur function, q-series.

Contents

1 Introduction

Recently, applications of the minor summation formula presented in [8] have been made in several directions, e.g., to study a certain limit law for

shifted Schur measures in [33], to find an explicit description of the skew-Capelli identity in [17], and to generalize further the so-called Littlewood formulas, for instance in [11] (see also [21], [13], [14]), etc. Moreover, the formula has been generalized to the case of hyperpfaffians in [23] (see also [22]).

In this paper, therefore, we treat again the minor summation formulas of Pfaffians and derive several basic formulas concerning Pfaffians from certain combinatorial theoretical points of view. In order to develop the study of these formulas nicely, we present also a Pfaffian version of the Lewis Carroll formula (Dodgson's identity) and of the Plücker relations. The proof and the discussion concerning these formulas have not been developed sufficiently in our previous papers because they are not directly related to the actual proofs in our applications of the minor summation formulas to obtaining various generating functions of the Schur functions, etc., whereas they have been studied from the early stage of the research.

One of the main purpose of the present paper is to prove the minor summation formulas using the lattice path method (see [31]) combined with the Lewis Carroll formula for Pfaffians. The proof thus obtained enables us to provide a combinatorial interpretation of the minor summation formula through lattice paths. The point is that the Lewis Carroll formula for Pfaffians plays efficiently to reduce the proof of the minor summation formulas comparing with a similar combinatorial discussion given in [31], and the present proof, consequently, may explain the meaning of the formulas more clearly.

The paper is organized as follows. In Section 2 we present two Pfaffian identities which may be called a Pfaffian version of the Lewis Carroll formula and the Plücker relations respectively, and in Section 3 we formulate the various type of minor summation formulas, where the Pfaffian analogue of the Lewis Carroll formula plays a key role at the derivation. In fact, we define the notion of the matrix formed by copfaffians which may sound abuse of languages, but we need to define a Pfaffian counterpart of the matrix of cofactors in the determinant theory. Then we can get a new expression of the minor summation formulas by employing these matrices of copfaffians. Without the notion of the matrices of copfaffians, it seems very hard to discover Gessel-Viennot type combinatorial proofs developed in Section 4, which simplify the lattice path proof of the minor summation formulas.

In Section 5 and Section 6 we give certain applications of the minor summation formulas to q-series. Actually, in Section 5 we show that Kawanaka's q-Littlewood identity is easily derived from the minor summation formulas, and in Section 6 we show that Kawanaka's q-Cauchy identity is proved by the Binet-Cauchy formula with some combinatorics. Furthermore, we shall give some variant of the Sundquist formula obtained in [32] which is considered as a two variable Pfafffian identity. We put this in the Appendix because the formula is not a direct application of the minor summation formula.

2 The Lewis Carroll formula and the Plücker relations

We provide a Pfaffian version of Lewis Carroll's formula (Dodgson's identity). We first recall the so-called Lewis Carroll formula, or known as Jacobi's formula which is an identity among the minor determinants. The

reader can find a restricted version of this identity and related topics in [1] and [28]. Furthermore, we present the Plücker's relations. The latter relations are also treated in [3], and in [18] they are called the (generalized) basic identity. We give a brief proof of ordinary Lewis Carroll's formula, which will be needed to establish the Pfaffian version, to make also this paper self-contained. We only use Cramer's formula to prove it.

Let us denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{Z}$ the set of integers. Let $[n]$ denote the set $\{1, 2, \dots, n\}$ for a positive integer n. For any finite set S and any nonnegative integer r, let $\binom{S}{r}$ denote the set of all r-element subsets of S. For example, $\binom{[n]}{r}$ stands for the set of all multi-indices $\{i_1, \dots, i_r\}$ such that $1 \leq i_1 < \cdots < i_r \leq n$. Let n, M and N be positive integers such that $n \leq M, N$ and let T be any M by N matrix. For any multi-indices $I = \{i_1, \dots, , i_n\} \in \binom{[M]}{n}$ and $J = \{j_1, \dots, j_n\} \in \binom{[N]}{n}$, let $T^I_J = T^{i_1 \dots i_n}_{j_1 \dots j_n}$ be the sub-matrix of T obtained by picking up the rows indexed by I and the columns indexed by J, i.e.,

$$T^I_J = \begin{pmatrix} t_{i_1 j_1} & \dots & t_{i_1 j_n} \\ \vdots & \ddots & \vdots \\ t_{i_n j_1} & \dots & t_{i_n j_n} \end{pmatrix}.$$

In the case of $n = M$ and $I = [M]$, we omit I from the above expression and write T_J for T^I_J, when there is no possibility of confusion. Similarly we may write T^I for T^I_J if n=N and $J = [N]$.

Though the notion of Pfaffians is less familiar than that of determinants, it is also well-known that the Pfaffian (of a skew-symmetric matrix) is expressed as a square root of the determinant of the corresponding matrix. We recall then first a more combinatorial definition of Pfaffians presented in [31]. Let $\mathfrak{S}_n$ be the symmetric group on the set of the letters $1, 2, \dots, n$, and for each permutation $\sigma \in \mathfrak{S}_n$ let $\operatorname{sgn}\sigma$ stand for $(-1)^{\ell(\sigma)}$, where $\ell(\sigma)$ denotes the number of inversions in σ.

In this paper we use the symbol $\{i_1, i_2, \dots i_r\}_<$ for the set $\{i_1, i_2, \dots i_r\}$ with the relation $i_1 < i_2 < \cdots < i_r$. Let $n = 2r$ be an even integer and let $A = (a_{ij})_{1 \leq i,j \leq n}$ be an n by n skew symmetric matrix (i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring. The *Pfaffian* $\operatorname{Pf}(A)$ of A is defined by

$$\operatorname{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)\, a_{\sigma_1 \sigma_2} \dots a_{\sigma_{n-1} \sigma_n}, \tag{1}$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{n-1}, \sigma_n\}_<\}$ of $[n]$ into 2-element blocks, and $\epsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix}.$$

For instance, when $n = 4$, the equation above reads:

$$\operatorname{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{14} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Note that a skew symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is determined by its upper triangular entries a_{ij} for $1 \leq i < j \leq n$.

A permutation $(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ which arises from a partition of $[n]$ into 2-element blocks is called a *perfect matching* or a 1-*factor*. We

say that the points σ_{2i-1} and σ_{2i} are *connected* to each other in this perfect matching σ. We can express a perfect matching graphically by arranging the lattice points 1, ..., n along the x-axis in the plane and representing the edges $(\sigma_{2i-1}, \sigma_{2i})$ by curves in the upper half plane. Two edges $(\sigma_{2i-1}, \sigma_{2i})$ and $(\sigma_{2j-1}, \sigma_{2j})$ in σ will be said to be *crossed* if the corresponding edges intersect in such an embedding. It is known that $\operatorname{sgn}\sigma$ agrees with $(-1)^k$ where k denotes the number of crossed pairs of edges in σ. We write $\mathcal{F}_n$ for the set of perfect matchings of $[n]$. For an example, the graphical representation of the perfect matching $\sigma = \{(1,4),(2,5),(3,6)\} \in \mathcal{F}_6$ is Figure 1 bellow, and its sign is -1.

For each $\pi \in \mathfrak{S}_n$, put $A^{\pi} = (a_{\pi(i)\pi(j)})$. From the definition above it is easy to see that

$$\operatorname{Pf}(A^{\pi}) = \operatorname{sgn}\pi \operatorname{Pf}(A). \tag{2}$$

It is a well-known fact that the following identities hold. For any skew symmetric $2n$ by $2n$ matrix A and any $2n$ by $2n$ matrix B we have

$$\begin{aligned} \operatorname{Pf}(A)^2 &= \det(A), \\ \operatorname{Pf}\left(BA{}^tB\right) &= \det(B)\operatorname{Pf}(A). \end{aligned} \tag{3}$$

The first identity is fundamental and we may use it implicitly hereafter. The reader can prove it by the exterior algebra, or can find a combinatorial proof in [31]. The second identity is a special case of Theorem 3.2.

Let a_{ij} be a fixed element of a given square matrix A, and denote by $(A; i, j)$ the square sub-matrix obtained by removing the ith row and jth column of A. That is to say, we can write $(A; i, j) = A^{\overline{\{i\}}}_{\overline{\{j\}}}$ in the above notation, where $\overline{I}$ stands for the complementary set of I. The determinant of $(A; i, j)$ is called a *minor* corresponding to a_{ij}, and the number $(-1)^{i+j}\det(A; j, i)$ is called a (i, j)-cofactor of A. Here the terminology "minor" is used in a wider sense. The cofactor matrix $\widetilde{A}$ of A is the matrix whose (i, j)-entry is the (i, j)-cofactor of A. Then the following theorem is due to Jacobi.

Theorem 2.1 *Let A be an n by n matrix and $\widetilde{A}$ be its cofactor matrix. Let $r \leq n$ and $I, J \subseteq [n]$, $\sharp I = \sharp J = r$. Then*

$$\det \widetilde{A}^I_J = (-1)^{|I|+|J|}(\det A)^{r-1} \det A^{\overline{J}}_{\overline{I}}, \tag{4}$$

where $\overline{I}, \overline{J} \subseteq [n]$ stand for the complements of I, J, respectively in $\subseteq [n]$. Here we denote $|I| = \sum_{i \in I} i$.

Proof. Let $\Delta(i, j) = (-1)^{i+j}\det(A; j, i)$ denote the (i, j)-cofactor of A. Then, by definition, the matrix of cofactors is

$$\widetilde{A} = \left(\Delta(i, j)\right) = \left((-1)^{i+j}\det(A; j, i)\right).$$

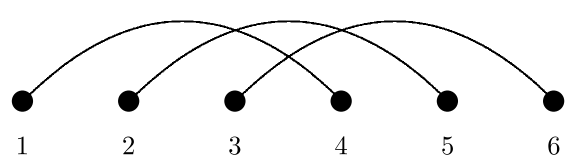

Figure 1: A perfect matching

Let $\overline{I} = \{p_1, \cdots, p_{n-r}\}$, $\overline{J} = \{q_1, \cdots, q_{n-r}\}$, and let $\sigma : \bar{I} \to \bar{J}$ denote the order-preserving bijection which maps p_k to q_k for $k = 1 \ldots n-r$. Set $M = (M_{ij})$ to be the matrix defined by

$$M_{ij} = \begin{cases} \Delta(i,j) & \text{if } i \in I, \\ \delta_{\sigma(i),j} & \text{if } i \in \overline{I}. \end{cases}$$

Then it is a direct simple algebra to see that the (i,j)-entry of the matrix $MA = B = (b_{ij})$ is

$$b_{ij} = \begin{cases} \delta_{ij} \det A & \text{if } i \in I, \\ a_{\sigma(i),j} & \text{if } i \in \overline{I}. \end{cases}$$

Accordingly we have $\det B = (\det A)^r \det A^{\overline{J}}_{\overline{I}}$. Meanwhile, it is easy to see that

$$\det M = (-1)^{\sum_{i \in \overline{I}}(i+\sigma(i))} \det \widetilde{A}^I_J = (-1)^{|\overline{I}|+|\overline{J}|} \det \widetilde{A}^I_J = (-1)^{|I|+|J|} \det \widetilde{A}^I_J$$

This proves the theorem. $\Box$

Example 2.2 *We put $I = J = \{1, n\} \subset [n]$ in the formula above and obtain the Desnanot-Jacobi adjoint matrix theorem:*

$$\det M \det M^{2,\ldots,n-1}_{2,\ldots,n-1} = \det M^{1,\ldots,n-1}_{1,\ldots,n-1} \det M^{2,\ldots,n}_{2,\ldots,n} - \det M^{1,\ldots,n-1}_{2,\ldots,n} \det M^{2,\ldots,n}_{1,\ldots,n-1},$$

which is also called Dodgson's formula (or the Lewis Carroll formula). For the details and the interesting story of the relations with the alternating sign matrices, see [1].

Let n be an even integer, and let A be a skew symmetric matrix of size n. For $1 \leq i < j \leq n$, let $(A; \{i,j\}, \{i,j\})$ denote the $(n-2)$ by $(n-2)$ skew symmetric sub-matrix obtained by removing both the ith and jth rows and both the ith and jth columns of A, i.e. $(A; \{i,j\}, \{i,j\}) = A^{\overline{\{i,j\}}}_{\overline{\{i,j\}}}$. Let us define $\gamma(i,j)$ by

$$\gamma(i,j) = (-1)^{i+j-1} \operatorname{Pf}(A; \{i,j\}, \{i,j\}) \tag{5}$$

for $1 \leq i < j \leq n$. We define the values of $\gamma(i,j)$ for $1 \leq j \leq i \leq n$ so that $\gamma(j,i) = -\gamma(i,j)$ always holds. Then the following expansion formula of Pfaffians along any row (resp. column) holds:

Proposition 2.3 *Let n be an even integer and $A = (a_{ij})$ be an n by n skew symmetric matrix. For any i, j we have*

$$\delta_{ij} \operatorname{Pf}(A) = \sum_{k=1}^{n} a_{kj} \gamma(k,i), \tag{6}$$

$$\delta_{ij} \operatorname{Pf}(A) = \sum_{k=1}^{n} a_{ik} \gamma(j,k). \tag{7}$$

Since a_{ij} and $\gamma(i,j)$ are skew symmetric, the reader sees immediately that the identities (6) and (7) are equivalent. Moreover, to prove the general case it is sufficient to show the case where $i = j = 1$ in view of the formula (2). This case can be proved combinatorially from the definition

(1) of Pfaffian. If we multiply the both sides of (6) by $\mathrm{Pf}(A)$ and use (3), then we obtain

$$\sum_{k=1}^{n} a_{ki}\gamma(k,j)\ \mathrm{Pf}(A) = \delta_{ij}\ [\mathrm{Pf}(A)]^2 = \delta_{ij}\ \det A.$$

Comparing this identity with the ordinary expansion of $\det A$, we obtain the following relation between $\Delta(i,j)$ and $\gamma(i,j)$:

$$\Delta(i,j) = \gamma(j,i)\ \mathrm{Pf}(A). \tag{8}$$

Definition 2.4 *Let n be an even integer. Given a skew symmetric matrix A of size n, let us call $\gamma(i,j)$ a* copfaffian *corresponding to a_{ij} (or (i,j)-*copfaffian*), and let $\widehat{A}$ denote the skew symmetric matrix whose (i,j)-entry is $\gamma(i,j)$, which we call the* copfaffian matrix *of A. Note that* (6) *and* (7) *implies*

$$^t\widehat{A}A = A\,^t\widehat{A} = \mathrm{Pf}(A)E_n, \tag{9}$$

where E_n denote the identity matrix of size n.

Example 2.5 *Let $P_n(s,t)$ denotes the skew symmetric matrix, whose (i,j)-entry is given by $s^{(i-1) \bmod 2 + j \bmod 2}t^{j-i-1}$ for $1 \le i < j \le n$, where $x \bmod 2$ stands for the remainder of x divided by 2. In Lemma 7 of [8], we proved the formula*

$$\mathrm{Pf}(x_i y_j)_{1\le i<j\le n} = \prod_{i=1}^{[n/2]} x_{2i-1} \prod_{j=1}^{[n/2]} y_{2j} \tag{10}$$

for an even integer n. From this formula, it is easy to see that the (i,j)-copfaffian of $P_n(s,t)$ is $(-1)^{j-i-1}s^{j-i-1}t^{(i-1) \bmod 2 + j \bmod 2}$. If $I = \{i_1, i_2, \dots, i_{2r-1}, i_{2r}\}_<$, then the formula (10) *also implies*

$$\mathrm{Pf}\left[P_n(s,t)_I^I\right] = s^{\sum_{k=1}^{2r}(i_k-k) \bmod 2}t^{\sum_{k=1}^{2r}(-1)^k i_k - r}. \tag{11}$$

The following result is considered as a Pfaffian version of Jacobi's formula.

Theorem 2.6 *Let n be an even integer, and let A be an n by n skew symmetric matrix. Then, for any $I \subseteq [n]$ such that $\sharp\, I = 2r$, we have*

$$\mathrm{Pf}\left[(\widehat{A})_I^I\right] = (-1)^{|I|-r}\,[\mathrm{Pf}(A)]^{r-1}\,\mathrm{Pf}(A_{\overline{I}}^{\overline{I}}). \tag{12}$$

In particular, we have $\widehat{\widehat{A}} = (\mathrm{Pf}\,A)^{m-2}A$ with $n = 2m$.

Proof. Let $\widetilde{A} = (\Delta(i,j))$ denote the matrix of the cofactors of A. From (8) we have $\widetilde{A} = \mathrm{Pf}(A)\,\widehat{A}$, thus $(\widetilde{A})_I^I = \mathrm{Pf}(A)\,(\widehat{A})_I^I$. It follows that

$$\det(\widetilde{A})_I^I = [\mathrm{Pf}(A)]^{2r}\ \det(\widehat{A})_I^I = (\det A)^r\ \det(\widehat{A})_I^I.$$

On the other hand, Theorem 2.1 implies that $\det(\widetilde{A}_I^I) = (\det A)^{2r-1}\det A_{\overline{I}}^{\overline{I}}$. Comparing these two identities, we obtain

$$\det(\widehat{A})_I^I = (\det A)^{r-1}\det A_{\overline{I}}^{\overline{I}}.$$

By taking the square root of both sides of this identity, we obtain

$$\mathrm{Pf}\left(\widehat{A}_I^I\right) = \pm\,[\mathrm{Pf}(A)]^{r-1}\,\mathrm{Pf}\left(A_{\overline{I}}^{\overline{I}}\right). \tag{13}$$

Next we need to show that the signature in (13) does not depend of A. Since the both sides of (13) are polynomials of the entries of A, their ratio $[\mathrm{Pf}(A)]^{r-1}\,\mathrm{Pf}\left(A_{\overline{I}}^{\overline{I}}\right)/\,\mathrm{Pf}\left(\widehat{A}_{I}^{I}\right)$ is a rational function of them. But this rational function can take only two values ± 1, it must be a constant, i.e. independent of the entries of A. To finish the proof we have to determine the sign. We substitute

$$S_n = P_n(1,1) = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 0 \end{pmatrix} \tag{14}$$

in the both sides of (13). Since $\widehat{S_n} = (\widehat{S}_{ij})$ with $\widehat{S}_{ij} = (-1)^{i+j-1}$ for $i < j$, Applying (11), we obtain $\mathrm{Pf}\,\widehat{S}_I = (-1)^{|I|-r}$ and $\mathrm{Pf}\,S_{\overline{I}} = 1$, which gives the desired sign. Let $f(A)$ and $g(A)$ denote the left and right hand side of (12). Then we have $(f+g)(f-g) = 0$ in the polynomial ring in the entries of A. Since $f(S_n) + g(S_n) \neq 0$, we conclude that $f = g$. This proves the theorem. $\square$

Given a skew symmetric matrix A, we write $A(i_1, i_2, \dots, i_{2k})$ for $A_{i_1,i_2,\dots,i_{2k}}^{i_1,i_2,\dots,i_{2k}}$.

Example 2.7 *Given a skew symmetric matrix A of size n, take $I = \{1,2,3,4\}$ in the theorem, then we obtain a formula which reads*

$$\begin{aligned} \mathrm{Pf}(A)\,\mathrm{Pf}\left(A(5,\dots,n)\right) = {} & \mathrm{Pf}\left(A(3,4,5,\dots,n)\right)\mathrm{Pf}\left(A(1,2,5,\dots,n)\right) \\ & - \mathrm{Pf}\left(A(2,4,5,\dots,n)\right)\mathrm{Pf}\left(A(1,3,5,\dots,n)\right) \\ & + \mathrm{Pf}\left(A(2,3,5,\dots,n)\right)\mathrm{Pf}\left(A(1,4,5,\dots,n)\right). \end{aligned}$$

This may be regarded as a Pfaffian version of Dodgson's identity given in Example 2.2.

We give some examples of the copfaffian matrices. Let $n = 2r$. If $S_n = (S_{ij})$ with $S_{ij} = 1$ for $i < j$, then we have $\widehat{S_n} = (\widehat{S}_{ij})$ with $\widehat{S}_{ij} = (-1)^{i+j-1}$ for $i < j$ as obtained in the above proof. Put $T_n = (T_{ij}) = P_n(0,1)$ and $\widehat{T_n} = \left(\widehat{T}_{ij}\right)$, then

$$T_{ij} = \begin{cases} 1 & \text{if } 1 \leq i < j, \text{ and } i \text{ and } j-i \text{ are both odd,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\widehat{T}_{ij} = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, we have

$$T_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \qquad \widehat{T}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let E_m denote the identity matrix of size m, and let $O_{m,n}$ denote the m by n zero matrix. When $m = n$, we simply write O_m for $O_{m,m}$. Let J_m denotes the symmetric matrix of size m defined by

$$J_m = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

We let $n = 2m$ and put $K_n = \begin{pmatrix} O_m & J_m \\ -J_m & O_m \end{pmatrix}$ and $L_n = \begin{pmatrix} O_m & E_m \\ -E_m & O_m \end{pmatrix}$. Then it is easy to see that ${}^tK_n = -K_n$, ${}^tL_n = -L_n$, $K_n^2 = L_n^2 = -I_n$, $\mathrm{Pf}(K_n) = 1$ and $\mathrm{Pf}(L_n) = (-1)^{\frac{m(m-1)}{2}}$. From Cramer's formula and (8), we have $\widehat{A} = \mathrm{Pf}(A)\,{}^tA^{-1}$ for a non-singular matrix A, which immediately implies $\widehat{K_n} = K_n$ and $\widehat{L_n} = (-1)^{m(m-1)/2} L_n$.

We next state a Pfaffian analogue of the Plücker relations (or known as the Grassmann-Plücker relations for determinants) and make a remark on a relation with the Lewis Carroll formula. It is an algebraic identity of degree two describing the relations among several subpfaffians. We point out here that this identity has been proved in [6] and in [3] in the framework of an exterior algebra.

Theorem 2.8 *Suppose m, n are odd integers. Let A be an $(m+n)\times(m+n)$ skew symmetric matrix. Fix a sequence of integers $I = \{i_1, i_2, \dots, i_m\}_< \subseteq [m+n]$ such that $\sharp I = m$. Denote the complement of I by $\overline{I} = \{k_1, k_2, \dots, k_n\}_< \subseteq [m+n]$ which has the cardinality n. Then the following relation holds.*

$$\sum_{j=1}^{m} (-1)^{j-1}\,\mathrm{Pf}\left(A_{I\setminus\{i_j\}}^{I\setminus\{i_j\}}\right)\mathrm{Pf}\left(A_{\{i_j\}\cup\overline{I}}^{\{i_j\}\cup\overline{I}}\right) = \sum_{j=1}^{n} (-1)^{j-1}\,\mathrm{Pf}\left(A_{I\cup\{k_j\}}^{I\cup\{k_j\}}\right)\mathrm{Pf}\left(A_{\overline{I}\setminus\{k_j\}}^{\overline{I}\setminus\{k_j\}}\right). \tag{15}$$

Proof. We only use the expansion formula of a Pfaffian given in Proposition 2.3. In fact, if we expand $\mathrm{Pf}(A_{\{i_j\}\cup\overline{I}}^{\{i_j\}\cup\overline{I}})$ along the i_jth row/column on the left-hand side and also expand $\mathrm{Pf}(A_{I\cup\{k_j\}}^{I\cup\{k_j\}})$ along the k_jth row/column on the right-hand side, and compare with each other, then it is immediate to see the desired equality. This identity is also proved directly from the definition (1) of a Pfaffian by using the notion of matching and related combinatorics. $\square$

The formula in the following assertion, which is called by the basic identity in [18], is regarded as a special case of the Plücker relations above.

Corollary 2.9 *Let A be a skew symmetric matrix of size N. Let $I = \{i_1, i_2, \dots, i_{2k}\}$ be a subset of $[N]$. Take an integer l which satisfies $2k + 2l \le N$. Then we have*

$$\begin{aligned} &\mathrm{Pf}(A(1,2,\dots,2l))\,\mathrm{Pf}(A(i_1,i_2,\dots,i_{2k},1,\dots,2l)) \\ &= \sum_{j=2}^{2k} (-1)^j\,\mathrm{Pf}(A(1,2,\dots,2l,i_1,i_j))\,\mathrm{Pf}(A(i_2,\dots,\widehat{i_j},\dots,i_{2k},1,\dots,2l)). \end{aligned} \tag{16}$$

Proof. Given a skew symmetric matrix $A = (a_{ij})_{1\le i,j\le N}$ of size N and a subset $I = \{i_1, i_2, \dots, i_{2k}\}$, we consider the skew symmetric matrix $B = (b_{ij})_{1\le i,j\le 2k+4l}$ of size $2k+4l$, whose (i,j)-entry b_{ij} is equal to $a_{p_ip_j}$, where the sequence $\{p_i\}_{i=1}^{2k+4l}$ is determined by

$$\begin{cases} p_\nu = \nu & \text{if } 1 \le \nu \le 2l, \\ p_{2l+\nu} = i_\nu & \text{if } 1 \le \nu \le 2k, \\ p_{2k+2l+\nu} = \nu & \text{if } 1 \le \nu \le 2l. \end{cases}$$

Now, apply Theorem 2.8 to B with $m = 2l+1$, $n = 2k+2l-1$, $I = \{1, 2, \dots, 2l+1\}$ and $\overline{I} = \{2l+2, 2l+3, \dots, 2k+4l\}$. Then, since each

summand on the left-hand side disappears except for the case $j = 2l - 1\,(i_j = i_1)$, the desired identity immediately follows from the identity in Theorem 2.8. $\square$

Remark 2.10 *If we put $k = 2$ in this corollary, then the identity is nothing but the identity in Example 2.7. This implies the basic identity partially covers the Lewis Carroll formula for Pfaffians.*

3 Summation formulas of Pfaffians

In this section we review the summation formulas of Pfaffians. We restate the theorems in [8] and, in the next section, we will give certain combinatorial proofs of Theorem 3.2, 3.4 and 3.5. In [8] we gave algebraic proofs of these theorems, whereas, in this paper we use the lattice paths combined with the Pfaffian version of Jacobi's formula (i.e. Theorem 2.6) to prove these theorems. Especially we will find that the Pfaffian version of Jacobi's formula is useful to simplify our lattice path proof, comparing with the combinatorial discussion given in [31], and give more insights to explain these formulas. Our proofs of the theorems described here will be postponed until the next section.

Lemma 3.1 *Let n, m and M be nonnegative integers, and let $N = 2N'$ be an even integer. Let A (resp. B) be a skew symmetric matrix of size M (resp. N) such that B is non-singular. Let T_{11}, T_{12}, T_{21} and T_{22} be an m by n, m by N, M by n and M by N rectangular matrix, respectively. Then*

$$\begin{aligned}
&\operatorname{Pf}(B)^{-1}\operatorname{Pf}\begin{pmatrix} T_{12}B{}^tT_{12} & T_{12}B{}^tT_{22} & T_{11}J_n \\ T_{22}B{}^tT_{12} & A+T_{22}B\,{}^tT_{22} & T_{21}J_n \\ -J_n{}^tT_{11} & -J_n{}^tT_{21} & O_n \end{pmatrix}\\
&=\operatorname{Pf}\begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{M,m} & A & T_{22}J_N & T_{21}J_n \\ -J_N{}^tT_{12} & -J_N{}^tT_{22} & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -J_n{}^tT_{11} & -J_n{}^tT_{21} & O_{n,N} & O_n \end{pmatrix}\\
&=\operatorname{Pf}\begin{pmatrix} J_M{}^tAJ_M & O_{M,m} & J_MT_{21} & J_MT_{22} \\ O_{m,M} & O_m & J_mT_{11} & J_mT_{12} \\ -{}^tT_{21}J_M & -{}^tT_{11}J_m & O_n & O_{n,N} \\ -{}^tT_{22}J_M & -{}^tT_{12}J_m & O_{N,n} & \frac{1}{\operatorname{Pf}(B)}\widehat{B} \end{pmatrix} \qquad (17)
\end{aligned}$$

Proof. The first identity follows from the following matrix identity

$$\begin{aligned}
&\begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{m,n} & A & T_{22}J_N & T_{21}J_n \\ -J_N{}^tT_{12} & -J_N{}^tT_{22} & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -J_n{}^tT_{11} & -J_n{}^tT_{21} & O_{n,N} & O_n \end{pmatrix}\begin{pmatrix} E_m & O_{m,M} & O_{m,N} & O_{m,n} \\ O_{M,m} & E_M & O_{M,N} & O_{M,n} \\ J_NB\,{}^tT_{12} & J_NB\,{}^tT_{22} & E_N & O_{N,n} \\ O_{n,m} & O_{n,M} & O_{n,N} & E_n \end{pmatrix}\\
&=\begin{pmatrix} T_{12}B\,{}^tT_{12} & T_{12}B\,{}^tT_{22} & T_{12}J_N & T_{11}J_n \\ T_{22}B\,{}^tT_{12} & A+T_{22}B\,{}^tT_{22} & T_{22}J_N & T_{21}J_n \\ O_{M,m} & O_{N,n} & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -J_n{}^tT_{11} & -J_n{}^tT_{21} & O_{n,N} & O_n \end{pmatrix},
\end{aligned}$$

which follows from (9). By taking the determinants of both sides of this equation and using $\operatorname{Pf}(J_N{}^t\widehat{B}J_N) = \operatorname{Pf}(\widehat{B}) = \operatorname{Pf}(B)^{-1}$ from (12), one

obtains

$$\mathrm{Pf}\begin{pmatrix} T_{12}B{}^t T_{12} & T_{12}B{}^t T_{22} & T_{11}J_n \\ T_{22}B{}^t T_{12} & A+T_{22}B\,{}^t T_{22} & T_{21}J_n \\ -J_n{}^t T_{11} & -J_n{}^t T_{21} & O_n \end{pmatrix} = \pm\,\mathrm{Pf}(B)\,\mathrm{Pf}\begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{M,m} & A & T_{22}J_N & T_{21}J_n \\ -J_N{}^t T_{12} & -J_N{}^t T_{22} & \frac{1}{\mathrm{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -J_n{}^t T_{11} & -J_n{}^t T_{21} & O_{n,N} & O_n \end{pmatrix}$$

Let $f(A,B,T_{11},T_{12},T_{21},T_{22})$ (resp. $g(A,B,T_{11},T_{12},T_{21},T_{22})$) denote the left-hand side (resp. the right-hand side) of this equation. Then we have $(f-g)(f+g)=0$ in the polynomial ring of the entries of A, B, T_{11}, T_{12}, T_{21}, T_{22}. Comparing the coefficients of the entries of A, we conclude that $f+g\neq 0$, which implies $f=g$. This shows that the signature does not depend on A, B and T_{ij}. The other identity is proved similarly. □

We restate Theorem 1 of [8] in the following form:

Theorem 3.2 *Let m and $N=2N'$ be even integers such that $m\leq N$. Let $T=(t_{ik})_{1\leq i\leq m,1\leq k\leq N}$ be an m by N rectangular matrix. Let $A=(a_{ij})_{1\leq i,j\leq N}$ be a non-singular skew-symmetric matrix of size N and let $\widehat{A}$ denote its copfaffian matrix. Then*

$$\sum_{I\in\binom{[N]}{m}} \mathrm{Pf}(A_I^I)\det(T_I)=\mathrm{Pf}(Q)$$

$$=\mathrm{Pf}(A)\,\mathrm{Pf}\begin{pmatrix} O_m & TJ_N \\ -J_N{}^t T & \frac{1}{\mathrm{Pf}(A)}J_N{}^t\widehat{A}J_N \end{pmatrix} = \mathrm{Pf}(A)\,\mathrm{Pf}\begin{pmatrix} O_m & J_mT \\ -{}^tTJ_m & \frac{1}{\mathrm{Pf}(A)}\widehat{A} \end{pmatrix}. \tag{18}$$

Here $Q=(Q_{ij})=TA\,{}^tT$, and its entries are given by

$$Q_{ij}=\sum_{1\leq k<l\leq N} a_{kl}\det(T_{kl}^{ij}), \qquad (1\leq i,j\leq m). \tag{19}$$

The first identity, i.e. $\sum \mathrm{Pf}(A_I^I)\det(T_I)=\mathrm{Pf}(Q)$, holds even if N is not even. When m is odd, we can immediately derive a similar formula from the case where m is even. So we only treat even cases in this paper. If we take $m=N=2r$ and $A=K_m$ in (18), then $\det(T)=\mathrm{Pf}(K_m)\det(T)=\mathrm{Pf}(T\,K_m{}^tT)$. This means that every determinant of even degree can be represented by a Pfaffian of the *same* degree.

When m and N are even integers such that $0\leq m\leq N$, and X and Y are m by N rectangular matrices, taking $A={}^tXK_mX$ and $T=Y$ in Theorem 3.2, we obtain the following corollary, which is the so-called Cauchy-Binet formula. For another proof which use Theorem 3.2 also, see [7].

Corollary 3.3 *Assume $m\leq N$, and let $X=(x_{ij})_{1\leq i\leq m,1\leq j\leq N}$ and $Y=(y_{ij})_{1\leq i\leq m,1\leq j\leq N}$ be any m by N matrices. Let $A=(a_{ij})_{1\leq i\leq N,1\leq j\leq N}$ be any N by N matrix. Then*

$$\sum_{K\in\binom{[N]}{m}} \det(X_K)\det(Y_K)=\det\left(X{}^tY\right). \tag{20}$$

Especially, using (20) twice, we obtain the following general version:

$$\sum_{I,J\in\binom{[N]}{m}} \det(A_J^I)\det(X_I)\det(Y_J)=\det\left(XA\,{}^tY\right). \tag{21}$$

The following theorem gives a minor summation formula, in which the index set I of a minor in the sum always includes some fixed column index set, say $\{1, 2, \ldots, n\}$. (See [8] and [31].)

Theorem 3.4 *Let m, n and N be positive integers such that $m-n$ and N are even and $0 \le m-n \le N$. Let $A = (a_{ij})_{1\le i,j\le N}$ be a nonsingular skew-symmetric matrix of size N. Let $T = (t_{ij})_{1\le i\le m, 1\le j\le n+N}$ be an m by $(n+N)$ rectangular matrix. Write the sets of column indices as $R^0 = \{1, \ldots, n\}$ and $R = \{n+1, \ldots, n+N\}$. Then*

$$\sum_{I \in \binom{R}{m-n}} \operatorname{Pf}(A_I^I) \det(T_{R^0 \uplus I}) = \operatorname{Pf}\begin{pmatrix} Q & T_{R^0} J_n \\ -J_n {}^t T_{R^0} & O_n \end{pmatrix}$$

$$= \operatorname{Pf}(A) \operatorname{Pf}\begin{pmatrix} O_m & T_R J_N & T_{R^0} J_n \\ -J_N {}^t T_R & \frac{1}{\operatorname{Pf}(A)} J_N {}^t \widehat{A} J_N & O_{N,n} \\ -J_n {}^t T_{R^0} & O_{n,N} & O_n \end{pmatrix}, \tag{22}$$

where Q is the m by m skew-symmetric matrix defined by $Q = T_R A\, {}^t T_R$, i.e.,

$$Q_{ij} = \sum_{1 \le k < l \le N} a_{kl} \det(T_{kl}^{ij}), \qquad (1 \le i, j \le m). \tag{23}$$

We shall later restate this theorem (and a proof) as Theorem 4.4 in a combinatorial description using lattice paths quite naturally.

The following theorem shows a minor summation formula for both the rows and columns.

Theorem 3.5 *Let M and N be even integers such that $M \le N$. Let $T = (t_{ij})_{1\le i\le M, 1\le j\le N}$ be any M by N rectangular matrix, and let $A = (a_{ij})_{1\le i,j\le M}$ (resp. $B = (b_{ij})_{1\le i,j\le N}$) be a nonsingular skew-symmetric matrix of size M (resp. size N). Then*

$$\sum_{r=0}^{\lfloor M/2 \rfloor} z^{2r} \sum_{I \in \binom{[M]}{2r}} \sum_{J \in \binom{[N]}{2r}} \operatorname{Pf}(A_I^I) \operatorname{Pf}(B_J^J) \det(T_J^I) = \operatorname{Pf}(A) \operatorname{Pf}\left[\frac{1}{\operatorname{Pf}(A)} \widehat{A} + z^2 Q\right]$$

$$= \operatorname{Pf}\begin{bmatrix} J_M {}^t A J_M & J_M \\ -J_M & z^2 Q \end{bmatrix} = \operatorname{Pf}(A) \operatorname{Pf}(B) \operatorname{Pf}\begin{bmatrix} \frac{1}{\operatorname{Pf}(A)} \widehat{A} & z T J_N \\ -z J_N {}^t T & \frac{1}{\operatorname{Pf}(B)} J_N {}^t \widehat{B} J_N \end{bmatrix}$$

$$= \operatorname{Pf}(A) \operatorname{Pf}(B) \operatorname{Pf}\begin{bmatrix} \frac{1}{\operatorname{Pf}(A)} J_n {}^t \widehat{A} J_n & z J_n T \\ -z {}^t T J_n & \frac{1}{\operatorname{Pf}(B)} \widehat{B} \end{bmatrix} \tag{24}$$

where $Q = T B\, {}^t T$ and $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x.

We also have the

Corollary 3.6 *Let M and N be nonnegative integers such that $M \le N$. Let $T = (t_{ij})$ be an M by N rectangular matrix. Let $B = (b_{ij})_{0\le i,j\le N}$ be a skew-symmetric matrix of size $(N+1)$.*

1. *If M is odd and $A = (a_{ij})_{0\le i,j\le M}$ is a nonsingular skew-symmetric*

matrix of size $(M+1)$*, then*

$$\sum_{r=0}^{\lfloor M/2\rfloor} z^{2r} \sum_{I\in\binom{[M]}{2r}} \sum_{J\in\binom{[N]}{2r}} \operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)\det(T_J^I)$$
$$+\sum_{r=0}^{\lfloor (M-1)/2\rfloor} z^{2r+1} \sum_{I\in\binom{[M]}{2r+1}} \sum_{J\in\binom{[N]}{2r+1}} \operatorname{Pf}(A_{\{0\}\cup I}^{\{0\}\cup I})\operatorname{Pf}(B_{\{0\}\cup J}^{\{0\}\cup J})\det(T_J^I)$$
$$=\operatorname{Pf}(A)\operatorname{Pf}\left(\frac{1}{\operatorname{Pf}(A)}\widehat{A}+Q\right), \qquad (25)$$

where $Q=(Q_{ij})_{0\le i,j\le M}$ *is given by*

$$Q_{ij}=\begin{cases} 0, & \text{if } i=j=0,\\ z\sum_{1\le k\le N} b_{0k}t_{jk}, & \text{if } i=0 \text{ and } 1\le j\le M,\\ z\sum_{1\le k\le N} b_{k0}t_{jk}, & \text{if } j=0 \text{ and } 1\le i\le M,\\ z^2\sum_{1\le k<l\le N} b_{kl}\det(T_{kl}^{ij}), & \text{if } 1\le i,j\le M. \end{cases} \qquad (26)$$

2. *If* M *is even and* $A=(a_{ij})_{0\le i,j\le M+1}$ *is a nonsingular skew-symmetric matrix of size* $(M+2)$*, then*

$$\sum_{r=0}^{\lfloor M/2\rfloor} z^{2r} \sum_{I\in\binom{[M]}{2r}} \sum_{J\in\binom{[N]}{2r}} \operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)\det(T_J^I)$$
$$+\sum_{r=0}^{\lfloor (M-1)/2\rfloor} z^{2r+1} \sum_{I\in\binom{[M]}{2r+1}} \sum_{J\in\binom{[N]}{2r+1}} \operatorname{Pf}(A_{\{0\}\cup I}^{\{0\}\cup I})\operatorname{Pf}(B_{\{0\}\cup J}^{\{0\}\cup J})\det(T_J^I)$$
$$=\operatorname{Pf}(A)\operatorname{Pf}\left(\frac{1}{\operatorname{Pf}(A)}\widehat{A}+Q\right), \qquad (27)$$

where $Q=(Q_{ij})_{0\le i,j\le M+1}$ *is given by*

$$Q_{ij}=\begin{cases} 0 & \text{if } i=j=0,\\ z\sum_{1\le k\le N} b_{0k}t_{jk}, & \text{if } i=0 \text{ and } 1\le j\le M,\\ z\sum_{1\le k\le N} b_{k0}t_{jk}, & \text{if } j=0 \text{ and } 1\le i\le M,\\ z^2\sum_{1\le k<l\le N} b_{kl}\det(T_{kl}^{ij}), & \text{if } 1\le i,j\le M,\\ 0 & \text{if } i=M+1 \text{ or } j=M+1. \end{cases} \qquad (28)$$

Theorem 3.7 *Let* m*,* n*,* M *and* N *be non-negative integers such that* M*,* N *and* $m-n$ *are even. We put* $R^0=\{1,\dots,m\}$*,* $S^0=\{1,\dots,n\}$*,* $R=\{m+1,\dots,m+M\}$ *and* $S=\{n+1,\dots,n+N\}$*. Let* $T=(t_{ij})_{1\le i\le m+M,1\le j\le n+N}$ *be any* $(m+M)$ *by* $(n+N)$ *rectangular matrix, and let* $A=(a_{ij})_{1\le i,j\le M}$ *(resp.* $B=(b_{ij})_{1\le i,j\le N}$*) be a nonsingular skew-symmetric matrix of size* M *(resp. size* N*). Then*

$$\sum_{\substack{\max(m,n)\le r\le\min(m+M,n+N)\\ r-\max(m,n)\text{ is even.}}} z^r \sum_{I\in\binom{R}{r-m}} \sum_{J\in\binom{S}{r-n}} \operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)\det(T_{S^0\uplus J}^{R^0\uplus I})$$
$$=\operatorname{Pf}(A)\operatorname{Pf}(B)\operatorname{Pf}\begin{pmatrix} O_m & O_{m,M} & zT_S^{R^0}J_N & zT_{S^0}^{R^0}J_n\\ O_{M,m} & \frac{1}{\operatorname{Pf}(A)}\widehat{A} & zT_S^RJ_N & zT_{S^0}^RJ_n\\ -zJ_N{}^tT_S^{R^0} & -zJ_N{}^tT_S^R & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n}\\ -zJ_n{}^tT_{S^0}^{R^0} & -zJ_n{}^tT_{S^0}^R & O_{n,N} & O_n \end{pmatrix} \qquad (29)$$

Corollary 3.8 *Let m, n, M and N be nonnegative integers such that $m-n$ is even and $M \le N$. We put $R^0 = \{1,\dots,m\}$, $S^0 = \{1,\dots,n\}$, $R = \{m+1,\dots,m+M\}$ and $S=\{n+1,\dots,n+N\}$. Let $T = (t_{ij})_{1\le i\le m+M, 1\le j\le n+N}$ be any $(m+M)$ by $(n+N)$ rectangular matrix and let $B = (b_{ij})_{1\le i,j\le N+1}$ be any skew-symmetric matrix of size $(N+1)$.*

1. *If M is odd and $A = (a_{ij})_{1\le i,j\le M+1}$ is a nonsingular skew-symmetric matrix of size $(M+1)$, then*

$$\sum_{\substack{\max(m,n)\le r\le \min(m+M,n+N)\\ r-\max(m,n) \text{ is even.}}} z^r \sum_{I\in\binom{R}{r-m}} \sum_{J\in\binom{S}{r-n}} \operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)\det(T_{S^0\uplus J}^{R^0\uplus I})$$
$$+\sum_{\substack{\max(m,n)\le r\le \min(m+M,n+N)\\ r-\max(m,n) \text{ is odd.}}} z^r \sum_{I\in\binom{R}{r-m}} \sum_{J\in\binom{S}{r-n}} \operatorname{Pf}(A_{I\uplus\{M+1\}}^{I\uplus\{M+1\}})\operatorname{Pf}(B_{J\uplus\{N+1\}}^{J\uplus\{N+1\}})\det(T_{S^0\uplus J}^{R^0\uplus I})$$
$$=\operatorname{Pf}(A)\operatorname{Pf}\begin{pmatrix} Q^{11} & Q^{12} & zT_{S^0}^{R^0}J_n \\ -{}^tQ^{12} & \frac{1}{\operatorname{Pf}(A)}\widehat{A}+Q^{22} & z\overline{T}_{S^0}^{R}J_n \\ -zJ_n{}^tT_{S^0}^{R^0} & -zJ_n{}^t\overline{T}_{S^0}^{R} & O_n\end{pmatrix}, \tag{30}$$

where $Q^{11} = (Q_{ij}^{11})_{1\le i,j\le m}$, $Q^{12} = (Q_{ij}^{12})_{1\le i\le m,\,1\le j\le M+1}$, $Q^{22} = (Q_{ij}^{22})_{1\le i,j\le M+1}$ is given by

$$Q_{ij}^{11} = z^2\sum_{1\le k<l\le N} b_{kl}\det(T_{n+k,n+l}^{ij}) \qquad \text{for } 1\le i,j\le m,$$

$$Q_{ij}^{12} = \begin{cases} z^2\sum_{1\le k<l\le N} b_{kl}\det(T_{n+k,n+l}^{i,j+m}) & \text{if } 1\le i\le m \text{ and } 1\le j\le M,\\ z\sum_{1\le k\le N} b_{k,N+1}T_{n+k}^{i} & \text{if } 1\le i\le m \text{ and } j=M+1,\end{cases}$$

$$Q_{ij}^{22} = \begin{cases} z^2\sum_{1\le k<l\le N} b_{kl}\det(T_{n+k,n+l}^{m+i,m+j}) & \text{if } 1\le i,j\le M,\\ z\sum_{1\le k\le N} b_{k,N+1}T_{n+k}^{m+i} & \text{if } 1\le i\le M \text{ and } j=M+1,\\ z\sum_{1\le k\le N} b_{N+1,k}T_{n+k}^{m+j} & \text{if } i=M+1 \text{ and } 1\le j\le M,\\ 0 & \text{if } i=j=M+1,\end{cases}$$

and $\overline{T}_{S^0}^{R}$ is the $(M+1)$ by n matrix in which its first M rows are the same as $T_{S^0}^{R}$ and the entries of its bottom row are all zero.

2. *If M is even and $A = (a_{ij})_{1\le i,j\le M+2}$ is a nonsingular skew-symmetric matrix of size $(M+2)$, then*

$$\sum_{\substack{\max(m,n)\le r\le \min(m+M,n+N)\\ r-\max(m,n) \text{ is even.}}} z^r \sum_{I\in\binom{R}{r-m}} \sum_{J\in\binom{S}{r-n}} \operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)\det(T_{S^0\uplus J}^{R^0\uplus I})$$
$$+\sum_{\substack{\max(m,n)\le r\le \min(m+M,m+N)\\ r-\max(m,n) \text{ is odd.}}} z^r \sum_{I\in\binom{R}{r-m}} \sum_{J\in\binom{S}{r-n}} \operatorname{Pf}(A_{I\uplus\{M+2\}}^{I\uplus\{M+2\}})\operatorname{Pf}(B_{J\uplus\{N+1\}}^{J\uplus\{N+1\}})\det(T_{S^0\uplus J}^{R^0\uplus I})$$
$$=\operatorname{Pf}(A)\operatorname{Pf}\begin{pmatrix} Q^{11} & Q^{12} & zT_{S^0}^{R^0}J_n \\ -{}^tQ^{12} & \frac{1}{\operatorname{Pf}(A)}\widehat{A}+Q^{22} & zT^{*R}_{S^0}J_n \\ -zJ_n{}^tT_{S^0}^{R^0} & -zJ_n{}^tT^{*R}_{S^0} & O_n\end{pmatrix}, \tag{31}$$

where $Q^{11} = (Q_{ij}^{11})_{1\le i,j\le m}$, $Q^{12} = (Q_{ij}^{12})_{1\le i\le m,\,1\le j\le M+2}$, $Q^{22} =$

$(Q_{ij}^{22})_{1\le i,j\le M+2}$ *is given by*

$$Q_{ij}^{11} = z^2 \sum_{1\le k<l\le N} b_{kl}\det(T_{n+k,n+l}^{ij}) \qquad \textit{for } 1\le i,j\le m,$$

$$Q_{ij}^{12} = \begin{cases} z^2 \sum_{1\le k<l\le N} b_{kl}\det(T_{n+k,n+l}^{i,j+m}) & \textit{if } 1\le i\le m \textit{ and } 1\le j\le M, \\ 0 & \textit{if } 1\le i\le m \textit{ and } j=M+1, \\ z\sum_{1\le k\le N} b_{k,N+1}T_{n+k}^{i} & \textit{if } 1\le i\le m \textit{ and } j=M+2, \end{cases}$$

$$Q_{ij}^{22} = \begin{cases} z^2 \sum_{1\le k<l\le N} b_{kl}\det(T_{n+k,n+l}^{m+i,m+j}) & \textit{if } 1\le i,j\le M, \\ z\sum_{1\le k\le N} b_{k,N+1}T_{n+k}^{m+i} & \textit{if } 1\le i\le M \textit{ and } j=M+2, \\ z\sum_{1\le k\le N} b_{N+1,k}T_{n+k}^{m+j} & \textit{if } i=M+2 \textit{ and } 1\le j\le M, \\ 0 & \textit{otherwise,} \end{cases}$$

and ${T^*}_{S^0}^{R}$ *is the* $(M+2)$ *by* n *matrix in which its first* M *rows are the same as* $T_{S^0}^{R}$ *and the entries of the last two rows are all zero.*

4 Proofs by Lattice Paths

In this section we give combinatorial proofs of the summation formulas of Pfaffians, i.e., Theorem 3.2, 3.4, 3.5, which are stated in Section 3. In [26] Okada gave this type of the formula related to a certain plane partition enumeration problem, but his formula was a very special case, that is, $A = S_N$, of ours. In [31] J.Stembridge gave a lattice path interpretation of this special summation formula, and gave proofs from this point of view. We follow his line in part and actually give a lattice path interpretation of our formulas and proofs from this viewpoint. However, it is important to notice here that the Pfaffian analogue of the Lewis Carroll formula (Theorem 2.6) makes possible the story clear. Thus, in this section, we provide an improved and a much simplified version of Stembridge's proof. We may say our proofs are closer to Gessel-Viennot's original proofs in [5] than those given in [31]. We note that Stembrige's proof [31] can be also generalized almost parallelly to proving Theorem 3.2, 3.4, 3.5, but we do not develop the proofs in this direction.

Now we review the basic terminology of lattice paths and fix notation. We follow the basic terminology in [5] and [31]. Let $D = (V, E)$ be an acyclic digraph without multiple edges. Further we assume that there are only finitely many directed paths between any two vertices. If u and v are any pair of vertices in D, let $\mathcal{P}(u, v)$ denote the set of all directed paths from u to v in D. Fix a positive integer n. An *n-vertex* is an n-tuple $\boldsymbol{v} = (v_1, \dots, v_n)$ of n vertices of D. Given any pair of n-vertices $\boldsymbol{u} = (u_1, \dots, u_n)$ and $\boldsymbol{v} = (v_1, \dots, v_n)$, an *$n$-path* from $\boldsymbol{u}$ to $\boldsymbol{v}$ is an n-tuple $\boldsymbol{P} = (P_1, \dots, P_n)$ of n paths such that $P_i \in \mathcal{P}(u_i, v_i)$. Let $\mathcal{P}(\boldsymbol{u}, \boldsymbol{v})$ denote the set of all n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$. Two directed paths P and Q will be said to be non-intersecting if they share no common vertex. An n-path $\boldsymbol{P}$ is said to be non-intersecting if P_i and P_j are non-intersecting for any $i \ne j$. Let $\mathcal{P}^0(\boldsymbol{u}, \boldsymbol{v})$ denote the subset of $\mathcal{P}(\boldsymbol{u}, \boldsymbol{v})$ which consists of all non-intersecting n-paths.

We assign a commutative indeterminate x_e to each edge e of D and call it the *weight* of the edge. Set the weight of a path P to be the product of the weights of its edges and denote it by $\mathrm{wt}(P)$. If u and v are any pair of vertices in D, define

$$h(u, v) = \sum_{P\in\mathcal{P}(u,v)} \mathrm{wt}(P).$$

The weight of an n-path is defined to be the product of the weights of its components. The sum of the weights of n-paths in $\mathcal{P}(\boldsymbol{u},\boldsymbol{v})$ (resp. $\mathcal{P}^0(\boldsymbol{u},\boldsymbol{v})$) is denoted by $F(\boldsymbol{u},\boldsymbol{v})$ (resp. $F^0(\boldsymbol{u},\boldsymbol{v})$).

Definition 4.1 *If $\boldsymbol{u}=(u_1,\dots,u_n)$ and $\boldsymbol{v}=(v_1,\dots,v_n)$ are n-vertices of D, then $\boldsymbol{u}$ is said to be D-compatible with $\boldsymbol{v}$ if every path $P\in\mathcal{P}(u_i,v_l)$ intersects with every path $Q\in\mathcal{P}(u_j,v_k)$ whenever $i<j$ and $k<l$.*

The following famous lemma is from [5]. We recall its proof here again to make not only this paper self-contained but also the subsequent discussion smooth.

Lemma 4.2 *(Lindström-Gessel-Viennot) Let $\boldsymbol{u}=(u_1,\dots,u_n)$ and $\boldsymbol{v}=(v_1,\dots,v_n)$ be two n-vertices in an acyclic digraph D. Then*

$$\sum_{\pi\in S_n}\operatorname{sgn}\pi\ F^0(\boldsymbol{u}^\pi,\boldsymbol{v})=\det[h(u_i,v_j)]_{1\le i,j\le n}. \tag{32}$$

Here, for any permutation $\pi\in S_n$, let $\boldsymbol{u}^\pi$ denote $(u_{\pi(1)},\dots,u_{\pi(n)})$. In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then

$$F^0(\boldsymbol{u},\boldsymbol{v})=\det[h(u_i,v_j)]_{1\le i,j\le n}. \tag{33}$$

Proof. From the definition of determinants we have

$$\det[h(u_i,v_j)]_{1\le i,j\le n}=\sum_{\pi\in\mathfrak{S}_n}\operatorname{sgn}(\pi)h(u_1,v_{\pi(1)})h(u_2,v_{\pi(2)})\dots h(u_r,v_{\pi(n)}). \tag{34}$$

For $\pi\in\mathfrak{S}_n$, let $P(\boldsymbol{u},\boldsymbol{v}^\pi)$ denote the set of all the n-paths $\boldsymbol{P}=\{P_1,\dots,P_n\}$ such that each path P_i connects u_i with $v_{\pi(i)}$ for $i=1,\dots,n$. Let $P^0(\boldsymbol{u},\boldsymbol{v}^\pi)$ denote the subset of $P(\boldsymbol{u},\boldsymbol{v}^\pi)$ which consists of all non-intersecting paths $\boldsymbol{P}\in P(\boldsymbol{u},\boldsymbol{v}^\pi)$. Let us define sets Π and Π^0 of configurations by

$$\begin{aligned}\Pi&=\left\{(\pi,\boldsymbol{P}):\pi\in\mathfrak{S}_n\text{ and }\boldsymbol{P}\in\mathcal{P}(\boldsymbol{u},\boldsymbol{v}^\pi))\right\},\\ \Pi^0&=\left\{(\pi,\boldsymbol{P}):\pi\in\mathfrak{S}_n\text{ and }\boldsymbol{P}\in\mathcal{P}^0(\boldsymbol{u},\boldsymbol{v}^\pi)\right\}.\end{aligned}$$

Then the right-hand side of (34) is the generating function of configurations $(\pi,\boldsymbol{P})\in\Pi$ with the weight $\operatorname{wt}(\pi,\boldsymbol{P})=\operatorname{sgn}(\pi)\operatorname{wt}(\boldsymbol{P})$. Now we describe an involution on the set $\Pi\setminus\Pi^0$ which reverse the sign of the associated weight. First fix an arbitrary total order on V. Let $C=(\pi,\boldsymbol{P})\in\Pi\setminus\Pi^0$. Among all vertices that occurs as intersecting points, let v denote the least vertex with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. Let $P_i(\to v)$ (resp. $P_i(v\to)$) denote the sub-path of P_i from u_i to v (resp. from v to $v_{\pi(i)}$). Set $C'=(\pi',\boldsymbol{P}')$ to be the configuration in which $P'_k=P_k$ for $k\ne i,j$,

$$P'_i=P_i(\to v)P_j(v\to),\qquad P'_j=P_j(\to v)P_i(v\to),$$

and $\pi'=\pi\circ(i,j)$. It is easy to see that $C'\in\Pi$ and $\operatorname{wt}(C')=-\operatorname{wt}(C)$. Thus $C\mapsto C'$ defines a sign reversing involution and, by this involution, one may cancel all of the terms $\{\operatorname{wt}(C):C\in\Pi\setminus\Pi^0\}$ and only the terms $\{\operatorname{wt}(C):C\in\Pi^0\}$ remains. Since $F^0(\boldsymbol{u}^\pi,\boldsymbol{v})=F^0(\boldsymbol{u},\boldsymbol{v}^{\pi^{-1}})$, we obtain the resulting identity. In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, the configurations $C\in\Pi^0$ occur only when $\pi=\mathrm{id}$, and are counted with the weight $\operatorname{wt}(P)$. This proves the lemma. □

Let n be an even integer and let $\boldsymbol{v} = (v_1, \dots, v_n)$ be an n-vertex. We write

$$\mathcal{F}(\boldsymbol{v}) = \{(v_{\sigma_1}, v_{\sigma_2}, \dots, v_{\sigma_{n-1}}, v_{\sigma_n}) : \sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n) \in \mathcal{F}_n\}$$

and call it the set of perfect matchings of $\boldsymbol{v}$.

Let $S = \{v_1 < \cdots < v_N\}$ be a finite totally ordered subset of V. We assume a commutative indeterminate $a_{v_i v_j}$ is assigned to each pair (v_i, v_j) $(i < j)$ of vertices in S. We write the assembly of the indeterminates as $A = (a_{v_i v_j})_{i<j}$. This upper triangular array uniquely defines a skew-symmetric matrix of size N, and we use the same symbol A to express this skew-symmetric matrix. Suppose m is even and $\boldsymbol{u} = (u_1, \dots, u_m)$ is an m-vertex. If $I = \{v_{i_1}, \dots, v_{i_m}\}_< \in \binom{S}{m}$ is an m-element subset of S, then we write A_I^I for $(a_{v_{i_k} v_{i_k}})_{1 \le k < l \le m}$. If $\boldsymbol{u} = (u_1, \dots, u_m)$ is an m-vertex and $\boldsymbol{v} = (v_1, \dots, v_n)$ (resp. $S = \{v_1, \dots, v_n\}_<$) is an n-vertex (resp. an n-element totally ordered subset of V), then let $H(\boldsymbol{u}, \boldsymbol{v})$ (resp. $H(\boldsymbol{u}, S)$) denote the m by n matrix $(h(u_i, v_j))_{1 \le i \le v,\, 1 \le j \le n}$. We consider the generating function of the set of non-intersecting m-paths from $\boldsymbol{u}$ to S weighted by subpfaffians of A:

$$Q(\boldsymbol{u}, S; A) = \sum_{I \in \binom{S}{m}} \operatorname{Pf}(A_I^I) F^0(\boldsymbol{u}, I)$$

The following theorem express this generating function by a Pfaffian, and is interpreted as a lattice path version of Theorem 3.2.

Theorem 4.3 *Let m and N be even integers such that $0 \le m \le N$. Let $\boldsymbol{u} = (u_1, \dots, u_m)$ be an m-vertex and $S = \{V_1 < \cdots < V_N\}$ be a totally ordered set of vertices in an acyclic digraph D. Let $A = (a_{V_i V_j})_{1 \le i < j \le N}$ be an skew-symmetric matrix with rows and columns indexed by S, and let $\widehat{A}$ denote its copfaffian matrix. Then we have*

$$\sum_{I \in \binom{S}{m}} \operatorname{Pf}(A_I^I) \sum_{\pi \in S_m} \operatorname{sgn} \pi \, F^0(\boldsymbol{u}^\pi, I) = \operatorname{Pf}(A) \operatorname{Pf} \begin{pmatrix} O_m & H(\boldsymbol{u}, S) J_N \\ -J_N \, {}^t H(\boldsymbol{u}, S) & \frac{1}{\operatorname{Pf}(A)} J_N {}^t \widehat{A} J_N \end{pmatrix}. \tag{35}$$

In particular, if $\boldsymbol{u}$ is D-compatible with S, then

$$\sum_{\substack{I \subseteq S \\ \sharp I = m}} \operatorname{Pf}(A_I^I) F^0(\boldsymbol{u}^\pi, I) = \operatorname{Pf}(A) \operatorname{Pf} \begin{pmatrix} O_m & H(\boldsymbol{u}, S) J_N \\ -J_N \, {}^t H(\boldsymbol{u}, S) & \frac{1}{\operatorname{Pf}(A)} J_N {}^t \widehat{A} J_N \end{pmatrix}. \tag{36}$$

Proof. Let $\widehat{a}_{V_i V_j}$ denote the (i,j)-copfaffian of A (i.e. $\widehat{A} = (\widehat{a}_{V_i V_j})$) and put $\alpha_{V_i V_j} = \frac{1}{\operatorname{Pf}(A)} \widehat{a}_{V_i V_j}$ for $1 \le i, j \le N$. Since multiplying J_N from right reverse the order of columns of $H(\boldsymbol{u}, S)$, we have

$$\operatorname{Pf} \begin{pmatrix} O_n & H(\boldsymbol{u}, S) J_N \\ -J_N \, {}^t H(\boldsymbol{u}, S) & \frac{1}{\operatorname{Pf}(A)} J_N {}^t \widehat{A} J_N \end{pmatrix} = \sum_\tau \operatorname{sgn} \tau \prod_{(u_i, V_j) \in \tau} h(u_i, V_j) \prod_{(V_k, V_l) \in \tau} \alpha_{V_k V_l}$$

summed over all perfect matchings τ on $(u_1, \dots, u_m, V_N, \dots, V_1)$ in which there are no edges connecting any two vertices of u. For an example of such a perfect matching, see Figure 2 bellow. We may interpret this Pfaffian as the generating function for all $(m+1)$-tuples $C = (\tau, P_1, \dots, P_m)$ such that $P_i \in \mathcal{P}(u_i, V_j)$ if there is an edge $(u_i, V_j) \in \tau$. This implies that

each vertex, say u_i, in $\boldsymbol{u}$ is always connected to a vertex, say V_j, in S, and remaining $(N-n)$ vertices of S are connected each other by edges which we write $(V_k, V_l) \in \tau$. The weight assigned to $C = (\tau, P_1, \dots, P_m)$ shall be $\operatorname{sgn}\tau \left(\prod_{(V_k,V_l)\in\tau} \alpha_{V_k V_l}\right) w(P_1)\cdots w(P_m)$. Let Σ denote the set of configurations $C = (\tau, P_1, \dots, P_m)$ satisfying the above condition, and let Σ^0 denote the subset consisting of all configurations $C = (\tau, P_1, \dots, P_m)$ such that $(P_1, \dots, P_m)$ is non-intersecting. We will show that there is a sign-reversing involution on $\Sigma \setminus \Sigma^0$, i.e. the set of the configurations $C = (\tau, P_1, \dots, P_m)$ with at least one pair of intersecting paths. Our proof here is essentially the same as that in Lemma 4.2. To describe the involution, first choose a fixed total order of the vertices, and consider an arbitrary configuration $C = (\tau, P_1, \dots, P_m) \in \Sigma \setminus \Sigma^0$. Among all vertices that occurs as intersecting points, let v denote the vertex which precedes all other points of intersections with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. We define $C' = (\tau', P'_1, \dots, P'_m)$ to be the configuration where $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\to v)P_j(v \to), \qquad P'_j = P_j(\to v)P_i(v \to),$$

and, if (u_i, V_k) and (u_j, V_l) are the edges of τ, then (u_i, V_l) and (u_j, V_k) are in τ' and all the other edges of τ' are the same as τ. Note that the multisets of edges appearing in $(P_1, \dots, P_m)$ and $(P'_1, \dots, P'_m)$ are identical, which means $\operatorname{wt}(C') = -\operatorname{wt}(C)$. Since this involution changes the sign of the associated weight, one may cancel all of the terms appearing in $\Sigma \setminus \Sigma^0$, aside from those with non-intersecting paths. For $C = (\tau, P_1, \dots, P_m) \in \Sigma^0$, let I denote the set of vertices of S connected to a vertex in $\boldsymbol{u}$, and let $\overline{I}$ denote the complementary set of I in S. Put $I = \{V_{i_1}, \dots, V_{i_m}\}_<$, then we can find a unique permutation $\pi \in S_m$ such that each $u_{\pi(k)}$ is connected to V_{i_k} in τ for $k = 1, \dots, m$. The remaining edges in τ which does not contribute to this permutation perform a perfect matching on $\overline{I}$ which we denote by σ. The $\operatorname{sgn}\tau$ is equal to $(-1)^{s(I,\overline{I})} \operatorname{sgn}\pi \operatorname{sgn}\sigma$, where $s(I, \overline{I})$ denote the shuffle number to merge I with $\overline{I}$ into S. Thus, if we put $m = 2m'$ and $N = 2N'$ for nonnegative integers m' and N', the sum of weights is equal to

$$\sum_I (-1)^{s(I,\overline{I})} \sum_{\pi\in S_m} \operatorname{sgn}\pi\, F^0(\boldsymbol{u}^\pi, I) \frac{1}{(\operatorname{Pf} A)^{N'-m'}} \operatorname{Pf}\left(\widehat{A}_{\overline{I}}^{\overline{I}}\right),$$

where I runs over all subsets of S of cardinality m. Theorem 2.6 implies

$$\operatorname{Pf}\left(\widehat{A}_{\overline{I}}^{\overline{I}}\right) = (-1)^{|\overline{I}|+N'-m'} \operatorname{Pf}(A)^{N'-m'-1} \operatorname{Pf}(A_I^I).$$

Since $I \cup \overline{I} = S$, we have $|I| + |\overline{I}| = \binom{N+1}{2} \equiv N' \pmod 2$. Meanwhile, it is easy to see $(-1)^{s(I,\overline{I})} = (-1)^{|I|-m'}$. This immediately implies (35). This completes the proof. $\square$

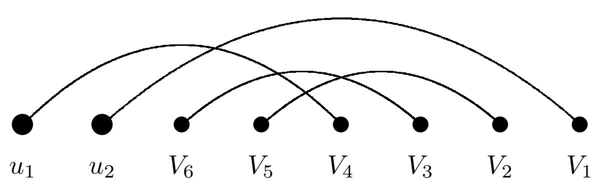

Figure 2: Proof of Theorem 4.3

In fact Theorem 4.3 is equivalent to Theorem 3.2. In [8] we gave an algebraic proof of Theorem 3.2 using the exterior algebra. One clearly sees that Theorem 4.3 is an easy consequence of Theorem 3.2 and Lemma 4.2. Here we give a proof that derives Theorem 3.2 from Theorem 4.3. Similarly one can derive Theorem 3.4 from Theorem 4.4, and also Theorem 3.5 from Theorem 4.5, but we will not give the details here and leave it to the reader.

Proof of Theorem 3.2. First we define a digraph D with vertex set $\mathbb{Z}^2$ and edges directed from u to v whenever $v-u=(1,0)$ or $(0,1)$. For $u=(i,j)$, we assign the weight x_j (resp. 1) to the edge with $v-u=(1,0)$ (resp. $(0,1)$). If $u=(i,1)$ and $v=(j,r)$, then $\lim_{r\to\infty} h(u,v)=h_{j-i}(x)$ is well-known as the complete symmetric function, which is defined by $\sum_{k\geq 0} h_k(x)t^k=\prod_{i\geq 1}\frac{1}{1-x_i t}$ (See [24]). Thus, if we fix constants $(\lambda_1,\cdots,\lambda_m)$ and $(\mu_1,\cdots,\mu_m)$ which satisfy $\lambda_1<\cdots<\lambda_m$ and $\mu_1<\cdots<\mu_m$, and take the vertices $u_i=(\lambda_i,1)$ and $v_i=(\mu_i,r)$ for $i=1,\cdots,m$, then, $\boldsymbol{u}$ and $\boldsymbol{v}$ are D-compatible, and from Lemma 4.2, we deduce

$$\lim_{r\to\infty} F^0(\boldsymbol{u},\boldsymbol{v})=\det\left(h_{\mu_j-\lambda_i}(x)\right)_{1\leq i,j\leq m}.$$

Let N be a positive integer such that $N\geq m\geq 0$ and $A=(a_{ij})$ be an N by N skew-symmetric matrix. We let $\boldsymbol{u}=(u_1,\cdots,u_m)$ with $u_i=(Ni,1)$ for $i=1\cdots m$ and $S=\{v_1,\cdots,v_N\}$ with $v_j=(j+Nm,r)$ for $j=1\cdots N$. Then, from Theorem 4.3 and by taking the limit $r\to\infty$, we obtain

$$\sum_{I=\{i_1<\cdots<i_m\}\subseteq[N]}\operatorname{Pf}(A_I^I)\det\left(h_{i_j+N(m-i)}(x)\right)_{1\leq i,k\leq m}=\operatorname{Pf}(Q)$$

where $Q=(Q_{ij})_{1\leq i,j\leq m}$ is given by

$$Q_{ij}=\sum_{1\leq k<l\leq N} a_{kl}\begin{vmatrix} h_{k+N(n-i)}(x) & h_{l+N(n-i)}(x)\\ h_{k+N(n-j)}(x) & h_{l+N(n-j)}(x)\end{vmatrix}$$

We use the fact that the $h_1,\cdots,h_{Nm}$ are algebraically independent over $\mathbb{Q}$ (See [24]). Thus we can replace each $h_{j+N(m-i)}$ with any commutative indeterminate t_{ij}, and we obtain Theorem 3.2. □

We next consider the lattice path version of Theorem 3.4. Let m, n and N be positive integers such that $m-n$ is even and $0\leq m-n\leq N$. Suppose that $S^0=\{v_1<\cdots<v_n\}$ is a fixed ordered list of vertices, and let $S=\{V_1<\cdots<V_N\}$ be a totally ordered set disjoint with S^0. For a subset I of S let $S^0\uplus I$ denote the union of S^0 and I, ordered so that each v_i precedes each $w\in I$. Let $A=(a_{V_iV_j})_{1\leq i,j\leq N}$ be a skew-symmetric matrix with rows and columns indexed by the totally ordered set S as before. We will obtain a formula of the generating function weighted by subpfaffians of A as follows:

$$Q(\boldsymbol{u};S^0,S;A)=\sum_{I\in\binom{S}{m-n}}\operatorname{Pf}(A_I^I)F^0(\boldsymbol{u},S^0\uplus I)$$

Theorem 4.4 *Let m, n and N be positive integers such that $m-n$ and N are even integers and $0\leq m-n\leq N$. Let $\boldsymbol{u}=(u_1,\cdots,u_m)$ be an m-vertex and $S^0=\{v_1<\cdots<v_n\}$ be an n-vertex in an acyclic digraph D. Let $S=\{V_1<\cdots<V_N\}$ be a finite totally ordered set of vertices*

which is disjoint with S_0. Let $A = (a_{V_iV_j})$ be an N by N skew-symmetric matrix with rows and columns indexed by S. Then

$$\sum_{I\in\binom{S}{m-n}} \mathrm{Pf}(A_I^I) \sum_{\pi\in S_m} \mathrm{sgn}\pi\, \boldsymbol{F}^0(u^\pi, S^0 \uplus I)$$
$$= \mathrm{Pf}(A)\,\mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u};S)J_N & H(\boldsymbol{u};S^0)J_n \\ -J_N\,{}^tH(\boldsymbol{u};S) & \frac{1}{\mathrm{Pf}(A)}J_N{}^t\widehat{A}J_N & O_{N,n} \\ -J_n\,{}^tH(\boldsymbol{u};S^0) & O_{n,N} & O_n \end{pmatrix} \tag{37}$$

where

$$H(\boldsymbol{u};S^0) = (h(u_i,v_j))_{1\le i\le m,1\le j\le n}\,,$$
$$H(\boldsymbol{u};S) = (h(u_i,V_j))_{1\le i\le m,1\le j\le N}\,.$$

In particular, if $\boldsymbol{u}$ is D-compatible with $I^0 \uplus S$, then we have

$$\sum_{I\in\binom{S}{m-n}} \mathrm{Pf}(A_I^I)\boldsymbol{F}^0(u, S^0 \uplus I)$$
$$= \mathrm{Pf}(A)\,\mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u};S)J_N & H(\boldsymbol{u};S^0)J_n \\ -J_N\,{}^tH(\boldsymbol{u};S) & \frac{1}{\mathrm{Pf}(A)}J_N{}^t\widehat{A}J_N & O_{N,n} \\ -J_n\,{}^tH(\boldsymbol{u};S^0) & O_{n,N} & O_n \end{pmatrix} \tag{38}$$

Proof. Let $\widehat{a}_{V_kV_l}$ denote the (k,l)-copfaffain of A, and put $\alpha_{V_kV_l} = \frac{1}{\mathrm{Pf}(A)}\widehat{a}_{V_kV_l}$ as before. We have

$$\mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u};S)J_N & H(\boldsymbol{u};S^0)J_n \\ -J_N{}^tH(\boldsymbol{u};S) & \frac{1}{\mathrm{Pf}(A)}J_N{}^t\widehat{A}J_N & O_{N,n} \\ -J_n{}^tH(\boldsymbol{u};S^0) & O_{n,N} & O_n \end{pmatrix}$$
$$= \sum_{\tau} \mathrm{sgn}\,\tau \prod_{(u_i,V_j)\in\tau} h(u_i,V_j) \prod_{(V_i,V_j)\in\tau} \alpha_{V_iV_j} \prod_{(u_i,V_j)\in\tau} h(u_i,V_j) \tag{39}$$

summed over all perfect matchings τ of $(u_1,\dots,u_m,V_N,\dots,V_1,v_n,\dots,v_1)$ in which there are no edges connecting any two vertices of $\boldsymbol{u}$, and each vertex in S^0 must be connected to a vertex in $\boldsymbol{u}$. An example of such a perfect matching is given below. This may be interpreted as the generating function for all $(m+1)$-tuples $C = (\tau, P_1, \dots, P_m)$ such that $P_i \in \mathcal{P}(u_i,v_j)$ if there is an edge $(u_i,v_j)\in\tau$, and $P_i \in \mathcal{P}(u_i,V_j)$ if there is an edge $(u_i,V_j)\in\tau$. The weight assigned to $C = (\tau, P_1,\dots,P_m)$ shall be $\mathrm{sgn}\,\tau \prod_{(V_k,V_l)\in\tau} \alpha_{V_kV_l} w(P_1)\dots w(P_m)$. We claim that the sign-reversing involution used in the previous proofs can be applied to this situation as well. In fact, quite the same arguments show that one may cancel all of the terms appearing in (39), aside from those with non-intersecting paths. In τ associated with these configuration $C = (\tau, P_1,\dots,P_m)$, each

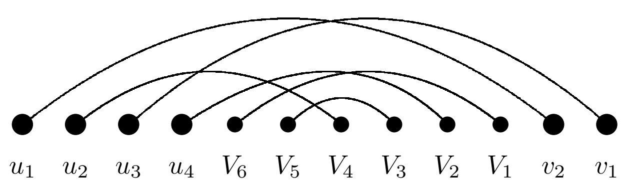

Figure 3: Proof of Theorem 4.4

v_k $(k=1,\dots,n)$ is always connected to a vertex in $\boldsymbol{u}$. This means that exactly n vertices of $\boldsymbol{u}$ are connected to vertices in S^0, and the remaining $(m-n)$ vertices are connected to certain vertices in S. Let I denote the set of vertices in S connected to vertices in $\boldsymbol{u}$, and let $\overline{I}$ denote its complementary set in S. Note that $\sharp I=(m-n)$ and $\sharp\overline{I}=(N-m+n)$. Let $S^0\uplus I$ denote the juxtaposition of vertices from I^0 and I arranged in this order, and, if we put $I^0\uplus I=(u_1^*,\dots,u_m^*)$ then there is a unique permutation $\pi\in S_m$ such that each $u_{\pi(k)}$ is connected to u_k^* for $k=1,\dots,m$. The remaining edges of τ whose both endpoints are included $\overline{I}$ define a unique perfect matching on $\overline{I}$. In this situation $\operatorname{sgn}\tau$ is equal to $s(I,\overline{I})\operatorname{sgn}\pi\ \operatorname{sgn}\sigma$. Thus, if we put $m-n=2m'$ and $N=2N'$ for nonnegative integers m' and N', then the sum of weights is equal to

$$\sum_I(-1)^{s(I,\overline{I})}\operatorname{sgn}\pi\,\boldsymbol{F}^0(u^\pi,I^0\uplus I)\frac{1}{(\operatorname{Pf}A)^{N'-m'}}\operatorname{Pf}(\widehat{A}_{\overline{I}}^{\overline{I}}),$$

where I runs over all subsets of S of cardinality $(m-n)$. From Theorem **??** we have

$$\operatorname{Pf}\left(\widehat{A}_{\overline{I}}^{\overline{I}}\right)=(-1)^{|\overline{I}|+N'-m'}(\operatorname{Pf}A)^{N'-m'-1}\operatorname{Pf}(A_I^I).$$

Since $I\cup\overline{I}=S$, we have $|I|+|\overline{I}|=\binom{N+1}{2}\equiv N'\pmod 2$. Meanwhile, it is easy to see $(-1)^{s(I,\overline{I})}=(-1)^{|I|-m'}$. This immediately implies (37). Lastly, if $\boldsymbol{u}$ is D-compatible with $I^0\uplus S$, then there is no non-intersecting path unless $\pi=id$, which immediately implies (38). This completes the proof. $\Box$

Now we show the following theorem for proving Theorem 3.5. For the purpose we consider a more general problem concerning with non-intersecting paths in which both starting points and end points vary.

Theorem 4.5 *Let M and N be even integers. Let $R=\{u_1<\dots<u_M\}$ and $S=\{v_1<\dots<v_N\}$ be totally ordered subsets of vertices in an acyclic digraph D. Let $A=(a_{u_iu_j})$ (resp. $B=(b_{v_iv_j})$ be a non-singular skew-symmetric matrix with rows and columns indexed by the vertices of R (resp. S). Then*

$$\sum_{\substack{0\le r\le\min(M,N)\\ r\ even}}z^r\sum_{I\in\binom{R}{r}}\sum_{J\in\binom{S}{r}}\operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)\sum_{\pi\in S_n}\operatorname{sgn}\pi\,F^0(I^\pi,J)$$
$$=\operatorname{Pf}(A)\operatorname{Pf}(B)\operatorname{Pf}\begin{pmatrix}\frac{1}{\operatorname{Pf}(A)}\widehat{A} & zH(R,S)J_N\\ -zJ_N\,{}^tH(R,S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N\end{pmatrix}. \tag{40}$$

In particular, if R is compatible with S, then

$$\sum_{\substack{0\le r\le\min(M,N)\\ r\ even}}z^r\sum_{I\in\binom{R}{r}}\sum_{J\in\binom{S}{r}}\operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)F^0(I,J)$$
$$=\operatorname{Pf}(A)\operatorname{Pf}(B)\operatorname{Pf}\begin{pmatrix}\frac{1}{\operatorname{Pf}(A)}\widehat{A} & zH(R,S)J_N\\ -zJ_N\,{}^tH(R,S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N\end{pmatrix}. \tag{41}$$

Proof. Let $\widehat{a}_{u_iu_j}$ (resp. $\widehat{b}_{v_iv_j}$) denote the (i,j)-copfaffain of A (resp. B). Put $\alpha_{u_iu_j}=\frac{1}{\operatorname{Pf}(A)}\widehat{a}_{u_iu_j}$ and $\beta_{v_iv_j}=\frac{1}{\operatorname{Pf}(B)}\widehat{b}_{v_iv_j}$. Then, we have

$$\operatorname{Pf}\begin{pmatrix}\frac{1}{\operatorname{Pf}(A)}\widehat{A} & zH(R,S)J_N\\ -zJ_N\,{}^tH(R,S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N\end{pmatrix}$$
$$=\sum_\tau\operatorname{sgn}\tau\prod_{(u_i,u_j)\in\tau}\alpha_{u_iu_j}\prod_{(u_i,v_j)\in\tau}zh(u_i,v_j)\prod_{(v_i,v_j)\in\tau}\beta_{v_iv_j}$$

summed over all perfect matchings τ on $(u_1,\dots,u_M,v_N,\dots,v_1)$. An example of such a perfect matching is Figure 4 bellow. As before we may interpret this Pfaffian as the generating function for all $(r+1)$-tuples $C=(\tau,P_1,\dots,P_r)$ which satisfies (i) r is an even integer such that $0\le r\le\min(M,N)$, (ii) there are exactly r edges whose one endpoint is in R and the other endpoint is S, and (iii) τ is a perfect matching on $(u_1,\dots,u_M,v_N,\dots,v_1)$ such that $P_i\in\mathcal{P}(u_i,v_j)$ if and only if there is an edge $(u_i,v_j)\in\tau$. The weight assigned to $C=(\tau,P_1,\dots,P_r)$ shall be

$$\operatorname{sgn}\tau\, z^r \prod_{(u_i,u_j)\in\tau}\alpha_{u_iu_j}\, w(P_1)\cdots w(P_r)\prod_{(v_i,v_j)\in\tau}\beta_{v_iv_j}.$$

The same argument as in the proof of Theorem 4.3 shows us that we can define a sign-reversing involution on the set of the configurations $C=(\tau,P_1,\dots,P_r)$ with at least one pair of intersecting paths, and this involution cancels all of the terms involving intersecting configurations of paths. Thus we need to sum over only non-intersecting configurations. Given a perfect matching τ on $(u_1,\dots,u_M,v_N,\dots,v_1)$ such that there are exactly r edges connecting a vertex in R and a vertex in S. Let I (resp. J) denote the subset of R (resp. S) which is composed of such endpoints of τ. Thus $\sharp I=\sharp J=r$, and r must be even. Let $\overline{I}$ (resp. $\overline{J}$) denote the complementary set of I (resp. J) in R (resp. S). Put $I=\{u_{i_1},\dots,u_{i_r}\}_<$ and $J=\{v_{j_1},\dots,v_{j_r}\}_<$, then there is a unique permutation π such that $u_{i_{\pi(\nu)}}$ is connected to v_{j_ν} in τ for $\nu=1,\dots,r$. If we put $M=2M'$, $N=2N'$ and $r=2r'$, then the sum of weights becomes

$$\sum_{\substack{r=0\\ r=2r'}}^{M}\sum_{\substack{I\subseteq R\\ \sharp I=r}}\sum_{\substack{J\subseteq S\\ \sharp J=r}}(-1)^{s(I,\overline{I})+s(J,\overline{J})}\sum_{\pi\in S_n}\operatorname{sgn}\pi\, z^r F^0(I^\pi,J)\frac{1}{\operatorname{Pf}(A)^{M'-r'}}\operatorname{Pf}(\widehat{A}^{\overline{I}}_{\overline{I}})\frac{1}{\operatorname{Pf}(B)^{N'-r'}}\operatorname{Pf}(\widehat{B}^{\overline{J}}_{\overline{J}}).$$

By (12) we have $\operatorname{Pf}(\widehat{A}^{\overline{I}}_{\overline{I}})=(-1)^{|\overline{I}|-M'+r'}\operatorname{Pf}(A)^{M'-r'-1}\operatorname{Pf}(A^I_I)$ and $\operatorname{Pf}(\widehat{B}^{\overline{J}}_{\overline{J}})=(-1)^{|J|-N'+r'}\operatorname{Pf}(B)^{N'-r'-1}\operatorname{Pf}(B^J_J)$. Further it is easy to see that $s(I,\overline{I})\equiv|I|-r' \pmod 2$ and $s(J,\overline{J})\equiv|J|-r' \pmod 2$. Since $I\cup\overline{I}=R$ and $J\cup\overline{J}=S$, we have $|I|+|\overline{I}|=\binom{M+1}{2}\equiv M' \pmod 2$ and $|J|+|\overline{J}|=\binom{N+1}{2}\equiv N' \pmod 2$. These identities immediately implies (40). $\square$

In the following theorem we assume D-compatibility of two regions to make our notation simple, but a more general theorem is also possible to establish.

Theorem 4.6 *Let m, n, M and N be nonnegative integers such that $M\equiv N\equiv m-n\equiv 0 \pmod 2$. Let $R^0=(u_1,\dots,u_m)$ (resp. $S^0=(v_1,\dots,v_n)$) be an m-vertex (resp. an n-vertex) in an acyclic digraph D. Let $R=\{U_1<\dots<U_M\}$ (resp. $S=\{V_1<\dots<V_N\}$) be totally ordered subsets of vertices in D which is disjoint with R^0 (resp. S^0). Assume that $R^0\uplus R$ is D-compatible with $S^0\uplus S$. Let $A=(a_{U_iU_j})$ (resp. $B=(b_{V_iV_j})$)*

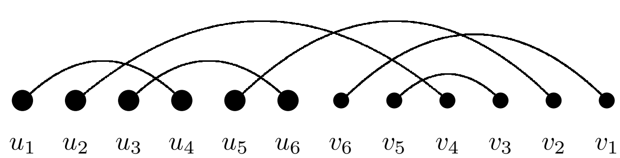

Figure 4: Proof of Theorem 4.5

be a skew-symmetric matrix with rows and columns indexed by the vertices of R (resp. S). Then

$$\sum_{\substack{\max(m,n)\le r\le\min(m+M,n+N)\\ r-\max(m,n)\text{ is even.}}} z^r \sum_{I\in\binom{R}{r-m}} \sum_{J\in\binom{S}{r-n}} \operatorname{Pf}(A_I^I)\operatorname{Pf}(B_J^J)F^0(R^0\uplus I, S^0\uplus J)$$
$$= \operatorname{Pf}(A)\operatorname{Pf}(B)\operatorname{Pf}\begin{pmatrix} O_m & O_{m,M} & zH(R^0,S)J_N & zH(R^0,S^0)J_n \\ O_{M,m} & \frac{1}{\operatorname{Pf}(A)}\widehat{A} & zH(R;S)J_N & zH(R;S^0)J_n \\ -zJ_N{}^tH(R^0,S) & -J_N{}^tzH(R;S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -zJ_n{}^tH(R^0,S^0) & -J_n{}^tzH(R;S^0) & O_{n,N} & O_n \end{pmatrix} \tag{42}$$

Proof. Without loss of generality we may assume that $m\ge n$. Let $\widehat{a}_{U_iU_j}$ (resp. $\widehat{b}_{V_iV_j}$) denote the (i,j)-copfaffain of A (resp. B). Put $\alpha_{U_iU_j}=\frac{1}{\operatorname{Pf}(A)}\widehat{a}_{U_iU_j}$ and $\beta_{V_iV_j}=\frac{1}{\operatorname{Pf}(B)}\widehat{b}_{V_iV_j}$. Further we put $R^0\uplus R=(u_1,\dots,u_m,U_1,\dots,U_M)=(u_1^*,\dots,u_{m+M}^*)$ and $S^0\uplus S=(v_1,\dots,v_n,V_1,\dots,V_N)=(v_1^*,\dots,v_{n+N}^*)$ for convenience. Then we have

$$\operatorname{Pf}\begin{pmatrix} O_m & O_{m,M} & zH(R^0,S)J_N & zH(R^0,S^0)J_n \\ O_{M,m} & \frac{1}{\operatorname{Pf}(A)}\widehat{A} & zH(R;S)J_N & zH(R;S^0)J_n \\ -zJ_N{}^tH(R^0,S) & -zJ_N{}^tH(R;S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -zJ_n{}^tH(R^0,S^0) & -zJ_n{}^tH(R;S^0) & O_{n,N} & O_n \end{pmatrix}$$
$$=\sum_{\tau}\operatorname{sgn}\tau\prod_{(U_k,U_l)\in\tau}\alpha_{U_kU_l}\prod_{(V_k,V_l)\in\tau}\beta_{V_kV_l}\prod_{(u_k^*,v_l^*)\in\tau}zh(u_k^*,v_l^*) \tag{43}$$

summed over all perfect matching on $(u_1^*,\dots,u_{m+M}^*,v_{n+N}^*,\dots,v_1^*)$. We can interpret this Pfaffian as the generating function for all $(r+1)$-tuples $C=(\tau,P_1,\dots,P_r)$ which satisfies (i) r is an integer such that $m\le r\le\min(m+M,n+N)$ and $r\equiv m\pmod 2$, (ii) τ is a perfect matching on $(u_1^*,\dots,u_{m+M}^*,v_{n+N}^*,\dots,v_1^*)=(u_1,\dots,u_m,U_1,\dots,U_M,V_N,\dots,V_1,v_n,\dots,v_1)$ such that $P_i\in\mathcal{P}(u_k^*,v_l^*)$ if and only if there is an edge $(u_k^*,v_l^*)\in\tau$ which is connecting a vertex in $R^0\uplus R$ and a vertex in $S^0\uplus S$. (iii) each vertex in R^0 must be connected to a vertex in $S^0\uplus S$, (iv) each vertex in S^0 must be connected to a vertex in $R^0\uplus R$, (v) and there are exactly r edges connecting a vertex in $R^0\uplus R$ with a vertex in $S^0\uplus S$. The weight assigned to $C=(\tau,P_1,\dots,P_r)$ shall be $\operatorname{sgn}\tau\, z^r\prod_{(U_k,U_l)\in\tau}\alpha_{U_kU_l}\prod_{(V_k,V_l)\in\tau}\beta_{V_kV_l}\operatorname{wt}(P_1)\cdots\operatorname{wt}(P_r)$. It is easy to see that the sign-reversing involution used in the previous proofs is applicable exactly as before, and we may cancel all the terms appearing in (43), aside from those with non-intersecting paths. Thus we only need to consider configurations $C=(\tau,P_1,\dots,P_r)$ with non-intersecting paths. From the assumption that $R^0\uplus R$ is D-compatible with $S^0\uplus S$, (a) each v_i, $i=1,\dots,n$, must be connected to u_i in τ and $P_i\in\mathcal{P}(u_i,v_i)$, (b) there are $r-m$ vertices $I=\{U_{i_1},\dots,U_{i_{r-m}}\}_{1\le i_1<\dots<i_{r-m}\le M}$ in R and $r-n$ vertices $J=\{V_{j_1},\dots,V_{j_{r-n}}\}_{1\le j_1<\dots<j_{r-n}\le N}$ such that each u_k, $k=n+1,\dots,m$, is connected to $V_{j_{k-n}}$ in τ and $P_k\in\mathcal{P}(u_k,V_{j_{k-n}})$ and each U_{i_k}, $k=1,\dots,r-m$, is connected to $V_{j_{k+m-n}}$ in τ and $P_{k+m}\in\mathcal{P}(U_{i_k},V_{j_{k+m-n}})$, (d) and the remaining $(m+M-r)$ vertices in R are connected each other in τ and the remaining $(n+N-r)$ vertices in S are connected each other in τ. We set $\overline{I}$ (resp. $\overline{J}$) to be the complementary set of I (resp. J) in R (resp. S). If we put $M=2M'$, $N=2N'$, $m-n=2l'$ and $r-m=2r'$ for nonnegative integers M', N', l' and r',

then the sum of the weights becomes

$$\sum_{r'=0}^{\min(M',N'-l')} \sum_{I\in\binom{R}{2r'}} \sum_{J\in\binom{S}{2(r'+l')}} (-1)^{s(I,\overline{I})+s(J,\overline{J})} \frac{1}{\operatorname{Pf}(A)^{M'-r'}\operatorname{Pf}(B)^{N'-r'-l'}} \operatorname{Pf}(\widehat{A}_{\overline{I}}^{\overline{I}}) \operatorname{Pf}(\widehat{B}_{\overline{J}}^{\overline{J}}) F^0(R^0 \uplus I, S^0 \uplus J).$$

By (12) we have $\operatorname{Pf}(\widehat{A}_{\overline{I}}^{\overline{I}}) = (-1)^{|\overline{I}|-M'+r'} \operatorname{Pf}(A)^{M'-r'-1} \operatorname{Pf}(A_I^I)$ and $\operatorname{Pf}(\widehat{B}_{\overline{J}}^{\overline{J}}) = (-1)^{|\overline{J}|-N'+l'+r'} \operatorname{Pf}(B)^{N'-l'-r'-1} \operatorname{Pf}(B_J^J)$. Further it is easy to see that $s(I,\overline{I}) \equiv |I| - r' \pmod 2$ and $s(J,\overline{J}) \equiv |J| - l' - r' \pmod 2$ Since $I \cup \overline{I} = R$ and $J \cup \overline{J} = S$, we have $|I| + |\overline{I}| = \binom{M+1}{2} \equiv M' \pmod 2$ and $|J| + |\overline{J}| = \binom{N+1}{2} \equiv N' \pmod 2$. These identities immediately imply (40). $\square$

5 Kawanaka's q-Littlewood formula

In [15], Kawanaka gave a certain q-series identity which is a generalization of the classical Schur-Littlewood identity (see [29]):

$$\sum_{\lambda} s_\lambda(x) = \prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j}$$

where the sum runs over all partitions λ. The Schur functions are well-known symmetric functions. The reader should consult [24] to see the detailed explanations of the symmetric functions. Here we only use a well-known determinant expression for the Schur functions. We use the notation in Macdonald's book [24]. For example, a *partition* is a non-increasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with finite non-zero parts. The number of non-zero parts are called the *length* and denoted by $\ell(\lambda)$. We assume the number of the variables is finite, say n, and $x = (x_1, \dots, x_n)$. Then the *Schur function* corresponding to a partition λ is defined to be

$$s_\lambda(x) = \frac{1}{\Delta(x)} \det(x_i^{\lambda_j+n-j}).$$

Here $\Delta(x) = \prod_{i<j}(x_i - x_j)$. In the following discussions we identify a partition with its Ferrers graph. Given a partition λ, the *hook-length* of λ at $\alpha = (i,j)$ is, by definition, $h(\alpha) = h(i,j) = \lambda_i + \lambda_j - i - j + 1$. Let $(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$ and $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$ for complex numbers a, q such that $|q| < 1$. We write $(a)_n$ (resp. $(a)_\infty$) for $(a;q)_n$ (resp. $(a;q)_\infty$) in short when there is no fear of confusion.

First of all, we recall Kawanaka's generalization of the Schur-Littlewood identity.

Theorem 5.1 *(Kawanaka)*

$$\sum_{\lambda} \prod_{\alpha\in\lambda} \frac{1+q^{h(\alpha)}}{1-q^{h(\alpha)}} s_\lambda(x) = \prod_{i=1}^{n} \frac{(-x_i q;q)_\infty}{(x_i;q)_\infty} \prod_{1\le i<j\le n} \frac{1}{1-x_i x_j} \tag{44}$$

where the sum runs over all partitions λ.

In this section we give a short proof of this identity as an application of the minor summation formula and then use this method to obtain a similar formula as follows.

Theorem 5.2

$$\sum_{\lambda} q^{n(\lambda)} \frac{\prod_{i=1}^{n}(a;q)_{\lambda_i+n-i}}{\prod_{\alpha\in\lambda}(1-q^{h(\alpha)})} s_\lambda(x) = \prod_{i=1}^{n-1}(a;q)_i \prod_{i=1}^{n} \frac{(aq^{n-1}x_i;q)_\infty}{(x_i;q)_\infty}. \quad (45)$$

Here $n(\lambda)=\sum_{i\geq 1}(i-1)\lambda_i$ *for a partition* λ.

(The referee pointed out the second theorem is a consequence of Cauchy's identity and the specialization of the Schur functions given in [24] ch. I l.3 ex.3). In order to prove the theorems, we first recall the q-binomial formula:

Lemma 5.3

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty}.\Box \quad (46)$$

The following lemma is a generalization of the q-binomial formula and becomes the key to the proof of Kawanaka's identity.

Lemma 5.4

$$\sum_{k,l\geq 0} \frac{(-q)_k(-q)_l}{(q)_k(q)_l} \frac{q^l-q^k}{q^k+q^l} x^k y^l = \frac{(-qx)_\infty}{(x)_\infty} \frac{(-qy)_\infty}{(y)_\infty} \frac{x-y}{1-xy}$$

Proof. Put

$$F(x,y) = \sum_{k,l\geq 0} \frac{(-q)_k(-q)_l}{(q)_k(q)_l} \frac{q^l-q^k}{q^k+q^l} x^k y^l = \sum_{k,l\geq 0} a_{kl} x^k y^l,$$

$$G(x,y) = \frac{x-y}{1-xy} \prod_{r=0}^{\infty} \frac{1+xq^{r+1}}{1-xq^r} \frac{1+yq^{r+1}}{1-yq^r} = \sum_{k,l\geq 0} b_{kl} x^k y^l.$$

Then

$$F(x,0) = \sum_{k\geq 1} \frac{(-q)_{k-1}}{(q)_{k-1}} x^k = x\frac{(-qx)_\infty}{(x)_\infty} = G(x,0).$$

Exactly the same argument leads to $F(0,y) = -y\frac{(-qy)_\infty}{(y)_\infty} = G(0,y)$, and This implies that $a_{k,0}=b_{k,0}$ and $a_{0,l}=b_{0,l}$ for $k,l\geq 0$. Next we claim that the coefficient of $x^k y^l$ of $(1-xy)F(x,y)$ is equal to the coefficient of $x^k y^l$ of $(1-xy)G(x,y)$ for $k,l\geq 1$. An easy calculation shows that

$$a_{kl} - a_{k-1,l-1} = 2(q^l-q^k)\frac{(-q)_{k-1}(-q)_{l-1}}{(q)_k(q)_l}.$$

On the other hand, the coefficient of $x^k y^l$ in $(1-xy)G(x,y) = (x-y)\frac{(-qx)_\infty(-qy)_\infty}{(x)_\infty(y)_\infty}$ is

$$\frac{(-q)_{k-1}}{(q)_{k-1}}\frac{(-q)_l}{(q)_l} - \frac{(-q)_k}{(q)_k}\frac{(-q)_{l-1}}{(q)_{l-1}} = 2(q^l-q^k)\frac{(-q)_{k-1}(-q)_{l-1}}{(q)_k(q)_l}.$$

This shows our claim holds, and in consequence we obtain $a_{kl}-a_{k-1,l-1} = b_{kl}-b_{k-1,l-1}$. This proves the lemma by induction. $\Box$

A key observation to prove Kawanaka's formula is the following identity which can be obtained from (1.7) of [24]:

$$\prod_{\alpha\in\lambda} \frac{1+q^{h(\alpha)}}{1-q^{h(\alpha)}} = \prod_{i=1}^{n} \frac{(-q)_{\ell_i}}{(q)_{\ell_i}} \prod_{i<j} \frac{1-q^{\ell_i-\ell_j}}{1+q^{\ell_i-\ell_j}}. \quad (47)$$

Here $\ell_i = \lambda_i + n - i$ with $\ell(\lambda) \leq n$. In fact, in ex. 1, ch. I, l.1 of [24], the following identity is shown:

$$\prod_{\alpha\in\lambda}(1-q^{h(\alpha)}) = \frac{\prod_{i=1}^{n}(q)_{\ell_i}}{\prod_{i<j}\left(1-q^{\ell_i-\ell_j}\right)}. \tag{48}$$

By the same argument one obtains

$$\prod_{\alpha\in\lambda}(1+q^{h(\alpha)}) = \frac{\prod_{i=1}^{n}(-q)_{\ell_i}}{\prod_{i<j}\left(1+q^{\ell_i-\ell_j}\right)}$$

from (1.7) of [24]. Taking the ratio of the equations one proves (47). We also use the following famous identities. (For the proof, see [31].)

Lemma 5.5 *Let n be an even integer. Let $x_1, \ldots, x_n$ be indeterminates. Then*

$$\mathrm{Pf}\left[\frac{x_i - x_j}{x_i + x_j}\right]_{1\leq i<j\leq n} = \prod_{1\leq i<j\leq n}\frac{x_i-x_j}{x_i+x_j}, \tag{49}$$

$$\mathrm{Pf}\left[\frac{x_i - x_j}{1 - x_i x_j}\right]_{1\leq i<j\leq n} = \prod_{1\leq i<j\leq n}\frac{x_i-x_j}{1-x_ix_j}. \quad \square \tag{50}$$

Let $A = (\alpha_{ij})_{i,j\geq 0}$ denote a skew symmetric matrix. As an application of Theorem 4.3 we obtain the following formula from the definition of the Schur functions.

Lemma 5.6 *Let n be an even integer. We denote by $s_\lambda(x)$ the Schur functions of n variables corresponding to a partition λ. Then*

$$\sum_{\lambda}\mathrm{Pf}(\alpha_{\ell_p\ell_q})_{1\leq p,q\leq n}\, s_\lambda(x) = \frac{1}{\Delta(x)}\mathrm{Pf}(\beta_{ij})_{1\leq i,j\leq n} \tag{51}$$

where λ runs all the partition such that $\ell(\lambda)\leq n$,

$$\beta_{ij} = \sum_{k,l\geq 0}\alpha_{kl}x_i^k x_j^l = \sum_{0\leq k<l}\alpha_{kl}\begin{vmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{vmatrix},$$

and $\Delta(x) = \prod_{1\leq i<j\leq n}(x_i - x_j)$.

Now we are in position to prove Kawanaka's formula.

Proof of Theorem 5.1. It is enough to prove the case where n is even. For a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ ($\lambda_1 \geq \cdots \geq \lambda_n \geq 0$), we put $\ell_i = \lambda_i + n - i$ ($i = 1, 2, \cdots, n$) as above. If we put $\alpha_{kl} = \frac{(-q)_k(-q)_l}{(q)_k(q)_l}\frac{q^l-q^k}{q^k+q^l}$ in (51), then (47) and (49) imply

$$\mathrm{Pf}\left[\alpha_{\ell_i\ell_j}\right]_{1\leq i,j\leq n} = \prod_{i=1}^{n}\frac{(-q)_{l_i}}{(q)_{l_i}}\prod_{1\leq i<j\leq n}\frac{q^{l_j}-q^{l_i}}{q^{l_i}+q^{l_j}} = \prod_{\alpha\in\lambda}\frac{1+q^{h(x)}}{1-q^{h(x)}}.$$

Thus, by Lemma 5.4 and (50) we obtain

$$\begin{aligned}\Delta(x)\sum_{\lambda}\prod_{\alpha\in\lambda}\frac{1+q^{h(x)}}{1-q^{h(x)}}s_\lambda(x) &= \mathrm{Pf}\left[\tfrac{(-qx_i)_\infty}{(x_i)_\infty}\tfrac{(-qx_j)_\infty}{(x_j)_\infty}\tfrac{x_i-x_j}{1-x_ix_j}\right]_{1\leq i<j\leq n} \\ &= \prod_{i=1}^{n}\frac{(-qx_i)_\infty}{(x_i)_\infty}\mathrm{Pf}\left[\tfrac{x_i-x_j}{1-x_ix_j}\right]_{1\leq i<j\leq n} \\ &= \Delta(x)\prod_{i=1}^{n}\frac{(-qx_i)_\infty}{(x_i)_\infty}\prod_{1\leq i<j\leq n}\frac{1}{1-x_ix_j}.\end{aligned}$$

This proves the theorem. □

As a formula similar to (49) and (50), the following is a Pfaffian version of the Vendermonde determinant.

Lemma 5.7 *Let $n = 2r$ be an even integer. Then*

$$\mathrm{Pf}\left[\frac{(x_i^r - x_j^r)^2}{x_i - x_j}\right]_{1\le i<j\le n} = \prod_{1\le i<j\le n}(x_i - x_j). \quad \square \tag{52}$$

This proof shows that replacing the entries of the Pfaffian by appropriate polynomials will be an interesting problem.

Proof of Theorem 5.2. Now we consider a skew symmetric matrix $A = (\alpha_{k,l})$ of size $n = 2r$ whose (k,l)-entry is defined by

$$\alpha_{k,l} = \frac{(a)_k(a)_l}{(q)_k(q)_l}\frac{(q^{rl}-q^{rk})^2}{q^l - q^k}.$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ we put $\ell_i = \lambda + n - i$. Then, from Lemma 5.7 we obtain

$$\begin{aligned}\mathrm{Pf}[\alpha_{\ell_i,\ell_j}]_{1\le i<j\le n} &= \prod_{i=1}^n \frac{(a)_i}{(q)_i}\prod_{1\le i<j\le n}(q^{\ell_j} - q^{\ell_i})\\ &= q^{\sum_{i=1}^n (i-1)(\lambda_i+n-i)}\prod_{i=1}^n\frac{(a)_i}{(q)_i}\prod_{1\le i<j\le n}(1 - q^{\ell_i-\ell_j})\\ &= \frac{q^{n(\lambda)+\frac{1}{6}n(n-1)(n-2)}\prod_{i=1}^n (a)_{\lambda_i+n-i}}{\prod_{c\in\lambda}(1-q^{h(c)})}.\end{aligned}$$

Now, to apply Lemma 5.6, we need to study the sum:

$$\begin{aligned}f_n(x,y) &= \sum_{k,l\ge 0}\frac{(a)_k(a)_l}{(q)_k(q)_l}\frac{(q^{rl}-q^{rk})^2}{q^l-q^k}x^k y^l,\\ &= \sum_{k,l\ge 0}\frac{(a)_k(a)_l}{(q)_k(q)_l}\left\{\sum_{\nu=1}^r q^{(\nu-1)k}q^{(n-\nu)l} - \sum_{\nu=1}^r q^{(n-\nu)k}q^{(\nu-1)l}\right\}x^k y^l,\\ &= \sum_{\nu=1}^r \frac{(aq^{\nu-1}x)_\infty}{(q^{\nu-1}x)_\infty}\frac{(aq^{n-\nu}y)_\infty}{(q^{n-\nu}y)_\infty} - \sum_{\nu=1}^r\frac{(aq^{n-\nu}x)_\infty}{(q^{n-\nu}x)_\infty}\frac{(aq^{\nu-1}y)_\infty}{(q^{\nu-1}y)_\infty}.\end{aligned}$$

Thus we have

$$f_n(x,y) = \frac{(aq^{n-1}x)_\infty(aq^{n-1}y)_\infty}{(x)_\infty(y)_\infty}g_n(x,y),$$

where $g_n(x,y)$ is a polynomial of x and y defined by

$$\begin{aligned}g_n(x,y) = &\sum_{\nu=1}^r\prod_{k=1}^{\nu-1}(1-q^{k-1}x)\prod_{k=\nu}^{n-1}(1-aq^{k-1}x)\prod_{k=1}^{n-\nu}(1-q^{k-1}y)\prod_{k=n-\nu+1}^{n-1}(1-aq^{k-1}y)\\ &-\sum_{\nu=1}^r\prod_{k=1}^{n-\nu}(1-q^{k-1}x)\prod_{k=n-\nu+1}^{n-1}(1-aq^{k-1}x)\prod_{k=1}^{\nu-1}(1-q^{k-1}y)\prod_{k=\nu}^{n-1}(1-aq^{k-1}y).\end{aligned}$$

By applying Lemma 5.6, it is not hard to see that in order to prove (45), it suffices to prove the following lemma, Lemma 6.8. □

Lemma 5.8 *Let n be even integer and let $g_n(x,y)$ be as above. Then we have*

$$\mathrm{Pf}[g_n(x_i,x_j)]_{1\le i<j\le n} = q^{\frac{1}{6}n(n-1)(n-2)} \prod_{k=1}^{n-1} (a)_k \prod_{1\le i<j\le n} (x_i - x_j).$$

Proof. The method we use here is quite similar to that of we used in the proof of Lemma 5.7. First of all the reader should notice that $g(x,y)$ is of degree $(n-1)$ as a polynomial in the variable x. This shows that $\mathrm{Pf}[g(x_i,x_j)]$ is a polynomial of degree at most $(n-1)$ if we see it as a polynomial of a fixed variable x_i. Since $g(x,y)$ is skew symmetric,i.e. $g(y,x) = -g(x,y)$, and this show that $(x_i - x_j)$ divides the Pfaffian, and, as before, the complete produce $\prod_{i<j}(x_i - x_j)$ must divide the Pfaffian. Thus we conclude that

$$\mathrm{Pf}[g(x_i,x_j)]_{1\le i<j\le n} = c \prod_{1\le i<j\le n} (x_i - x_j).$$

If we see the right-hand side as a polynomial of a fixed variable x_i, then it is of degree $(n-1)$, and this shows the constant c must not include x_i. Now, to determine the constant c, which is independent of x_i, we compare the coefficient of the monomial $\prod_{i=1}^{n} x_i^{n-i}$ of the both sides. First we consider the left-hand side. The Pfaffian is the sum of polynomials $\mathrm{sgn}\,\sigma g_n(x_{i_1},x_{j_1})\cdots g_n(x_{i_r},x_{j_r})$ for all perfect matching $\sigma = ((i_1,j_1),\dots,(i_r,j_r))$ of $[n]$, The monomial which contributes to the monomial $\prod_{i=1}^{n} x_i^{n-i}$ in the polynomial $g_n(x_{i_k},x_{j_k})$ is $x_{i_k}^{n-i_k} x_{j_k}^{n-j_k}$. This shows hence that the coefficient of $\prod_{i=1}^{n} x_i^{n-i}$ in the left-hand side Pfaffian is equal to $\mathrm{Pf}\left[[x^{n-i}y^{n-j}]g_n(x,y)\right]$. Thus, to prove the desired identity it suffices to prove the identity

$$\mathrm{Pf}\left[[x^{n-i}y^{n-j}]g_n(x,y)\right]_{1\le i<j\le n} = q^{\frac{1}{6}n(n-1)(n-2)} \prod_{k=1}^{n-1} (a)_k.$$

Here $[x^a y^b]f(x,y)$ stands for the coefficient of the $x^a y^b$ in the polynomial $f(x,y)$. Since the determination of the sign of the Pfaffain is easy, to prove the identity it suffices to show the following lemma. □

Lemma 5.9 *Let $n = 2r$ be an even integer and let $h_n(x,y) = h_n(a,b,q,t;x,y)$ be the polynomial of x and y defined by*

$$\sum_{\nu=1}^{r} \prod_{k=1}^{\nu-1} (1-q^{k-1}x) \prod_{k=\nu}^{n-1} (1-aq^{k-1}x) \prod_{k=1}^{n-\nu} (1-t^{k-1}y) \prod_{k=n-\nu+1}^{n-1} (1-bt^{k-1}y)$$
$$-\sum_{\nu=1}^{r} \prod_{k=1}^{n-\nu} (1-q^{k-1}x) \prod_{k=n-\nu+1}^{n-1} (1-aq^{k-1}x) \prod_{k=1}^{\nu-1} (1-t^{k-1}y) \prod_{k=\nu}^{n-1} (1-bt^{k-1}y).$$

Then we have

$$\det\left[[x^{n-i}y^{n-j}]h_n(x,y)\right]_{1\le i,j\le n} = (qt)^{\frac{1}{6}n(n-1)(n-2)} \prod_{i=1}^{n-1} (a;q)_i \prod_{j=1}^{n-1} (b;t)_j.$$

Proof. Note that $[x^{n-i}y^{n-j}]h_n(x,y)$ has degree $(n-1)$ regarded as a polynomial in either a or b, This means the determinant has degree at most $n(n-1)$ in either variable a or b. We claim that $\prod_{i=1}^{n-1}(a;q)_i \prod_{j=1}^{n-1}(b;t)_j$ divides the determinant. For this purpose we want to show that $(1-aq^{i-1})$

divides the determinant $(n-i)$ times for each $i = 1, \dots, (n-1)$. This can be done by substituting $a = q^{-k}$ $(k = 0, \dots, n-2)$ into the determinant and computing the rank. The details are left to the reader. By this argument, one see that

$$\det \Big[[x^{n-i} y^{n-j}] h_n(x,y) \Big]_{1 \le i,j \le n} = c \prod_{i=1}^{n-1} (a;q)_i \prod_{j=1}^{n-1} (b;t)_j,$$

where c is a constant independent of a and b. To find c, Compare the constant term of the both sides regarding them as polynomials of a and b. □

6 Kawanaka's q-Cauchy identity

In [16] Kawanaka gave a q-Cauchy formula, which is regarded as a determinant version of Kawanaka's q-Littlewood formula in the previous section. Before we state the theorem we need some definitions. Let λ and μ be partitions, and let $c = (i,j)$ be any cell in the plane. As a natural generalization of the ordinary hook length $h_\lambda(c) = \lambda_i + \lambda'_j - i - j + 1$, Kawanaka introduced a new statistic

$$h_{\lambda\mu}(c) = \lambda_i + \mu'_j - i - j + 1$$

in [16]. For example, let $\lambda = (4,3,1,1)$ and $\mu = (3,3)$. If we fill each cell c of λ with the numbers $h_{\lambda\mu}(c)$, then it looks as follows:

$$\begin{array}{|c|c|c|c|} \hline 5 & 4 & 3 & 0 \\ \hline 3 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 0 & \multicolumn{3}{c}{} \\ \cline{1-1} -1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

In [16], he also defined

$$n(\lambda,\mu) = \sum_{(i,j) \in \lambda - \mu} (\lambda'_j - i) = \sum_{(i,j) \in \lambda - \mu} (i - \mu'_j - 1)$$

which is regarded as a generalization of $n(\lambda) = \sum_{i \ge 1} (i-1)\lambda_i$ in [24]. Let t be an indeterminate. For any partitions λ and μ, define a rational function $J_{\lambda\mu}(t)$ in t by

$$J_{\lambda\mu}(t) = t^{n(\lambda,\mu)} \prod_{c \in \lambda} \frac{1 + t^{h_{\lambda\mu}(c)}}{1 - t^{h_\lambda(c)}} \cdot t^{n(\mu,\lambda)} \prod_{c \in \mu} \frac{1 + t^{h_{\mu\lambda}(c)}}{1 - t^{h_\mu(c)}}.$$

Theorem 6.1 *(Kawanaka) Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two independent sequences of variables. For a partition λ let $s_\lambda(x)$ and $s_\lambda(y)$ be the corresponding Schur functions in x and y respectively. Then we have the following identity:*

$$\begin{aligned} &\sum_{\lambda,\mu} q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2) s_\lambda(x) s_\mu(y) \\ &= \prod_{i \ge 1} \frac{(-qx_i;q^2)_\infty}{(qx_i;q^2)_\infty} \prod_{j \ge 1} \frac{(-qy_j;q^2)_\infty}{(qy_j;q^2)_\infty} \prod_{i,j \ge 1} \frac{1}{1 - x_i y_j}. \end{aligned} \tag{53}$$

Here $|\lambda-\mu|$ is the number of the cells in the set-theoretical difference $\{c : c\in\lambda, c\notin\mu\}$ and $|\mu-\lambda|$ is the number of the cells in $\{c : c\notin\lambda, c\in\mu\}$.

Lemma 6.2

$$\sum_{k,l\geq 0}\frac{(-q^2;q^2)_k}{(q^2;q^2)_k}\frac{(-q^2;q^2)_l}{(q^2;q^2)_l}\frac{2x^ky^l}{q^{k-l}+q^{l-k}}=\frac{(-qx;q^2)_\infty}{(qx;q^2)_\infty}\frac{(-qy;q^2)_\infty}{(qy;q^2)_\infty}\frac{1}{1-xy}. \tag{54}$$

Proof. Put

$$F(x,y)=\sum_{k,l\geq 0}\frac{(-q^2;q^2)_k}{(q^2;q^2)_k}\frac{(-q^2;q^2)_l}{(q^2;q^2)_l}\frac{2x^ky^l}{q^{k-l}+q^{l-k}}=\sum_{k,l\geq 0}a_{kl}x^ky^l,$$

$$G(x,y)=\frac{(-qx;q^2)_\infty}{(qx;q^2)_\infty}\frac{(-qy;q^2)_\infty}{(qy;q^2)_\infty}\frac{1}{1-xy}.$$

First, we compare the coefficients of x^ky^l in $(1-xy)F(x,y)$ and $(1-xy)G(x,y)$. By Lemma 5.3 we have

$$(1-xy)G(x,y)=\sum_{k,l\geq 0}\frac{(-1;q^2)_k(-1;q^2)_l}{(q^2;q^2)_k(q^2;q^2)_l}q^{k+l}x^ky^l.$$

Meanwhile, the coefficient of x^ky^l in $(1-xy)F(x,y)$ for $k,l\geq 1$ is equal to

$$\begin{aligned}&a_{kl}-a_{k-1,l-1}\\&=\frac{(-q^2;q^2)_{k-1}(-q^2;q^2)_{l-1}}{(q^2;q^2)_k(q^2;q^2)_l}\{(1+q^{2k})(1+q^{2l})-(1-q^{2k})(1-q^{2l})\}\frac{2q^kq^l}{q^{2k}+q^{2l}}\\&=4\frac{(-q^2;q^2)_{k-1}(-q^2;q^2)_{l-1}}{(q^2;q^2)_k(q^2;q^2)_l}q^{k+l}=\frac{(-1;q^2)_k(-1;q^2)_l}{(q^2;q^2)_k(q^2;q^2)_l}q^{k+l}.\end{aligned}$$

When $l=0$, it is easy to see that the coefficient of x^k in $(1-xy)F(x,y)$ is equal to

$$\frac{(-q^2;q^2)_k}{(q^2;q^2)_k}\frac{2q^k}{1+q^{2k}}=\frac{(-1;q^2)_k}{(q^2;q^2)_k}q^k.$$

On the other hand, when $=0$, it is also easy to see that the coefficient of y^l in $(1-xy)F(x,y)$ is equal to $\frac{(-1;q^2)_l}{(q^2;q^2)_l}q^l$. Thus, the coefficients agree in all cases, and we conclude that $(1-xy)F(x,y)=(1-xy)G(x,y)$. This completes the proof. $\square$

The following identities are known as the Cauchy determinants (see [24]).

Proposition 6.3

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\leq i,j\leq n}=\frac{\prod_{1\leq i<j\leq n}(x_i-x_j)\prod_{1\leq i<j\leq n}(y_i-y_j)}{\prod_{1\leq i,j\leq n}(x_i+y_j)}, \tag{55}$$

$$\det\left(\frac{1}{1-x_iy_j}\right)_{1\leq i,j\leq n}=\frac{\prod_{1\leq i<j\leq n}(x_i-x_j)\prod_{1\leq i<j\leq n}(y_i-y_j)}{\prod_{1\leq i,j\leq n}(1-x_iy_j)}. \tag{56}$$

Lemma 6.4 *Let $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_n)$ be partitions such that $\ell(\lambda),\ell(\mu)\leq n$. We put $k_i=\lambda_i+n-i$ and $\ell_i=\mu_i+n-i$ for $1\leq i\leq n$. If we put*

$$a_{kl}=\frac{(-q^2;q^2)_k}{(q^2;q^2)_k}\frac{(-q^2;q^2)_l}{(q^2;q^2)_l}\frac{2q^{k+l}}{q^{2k}+q^{2l}},$$

then

$$\det(a_{k_i\ell_j})_{1\le i,j\le n} = q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2). \tag{57}$$

Proof. From (55) we obtain

$$\begin{aligned}
&\det(a_{k_i\ell_j})_{1\le i,j\le n}\\
&= 2^n q^{\sum_i(2i-1)k_i+\sum_j(2j-1)\ell_j} \frac{\prod_{i<j}(1-q^{2(k_i-k_j)})}{\prod_i (q^2;q^2)_{k_i}}\cdot\\
&\qquad\times \frac{\prod_{i<j}(1-q^{2(\ell_i-\ell_j)})}{\prod_j (q^2;q^2)_{\ell_j}}\cdot \frac{\prod_i(-q^2;q^2)_{k_i}\prod_j(-q^2;q^2)_{\ell_j}}{\prod_{i,j}(q^{2k_i}+q^{2\ell_j})}.
\end{aligned}$$

By (48) we have

$$\begin{aligned}
\prod_{c\in\lambda}\frac{1}{1-q^{2h_\lambda(c)}} &= \frac{\prod_{1\le i<j\le n}(1-q^{2(k_i-k_j)})}{\prod_{i=1}^n (q^2;q^2)_{k_i}},\\
\prod_{c\in\lambda}\frac{1}{1-q^{2h_\mu(c)}} &= \frac{\prod_{1\le i<j\le n}(1-q^{2(\ell_i-\ell_j)})}{\prod_{j=1}^n (q^2;q^2)_{\ell_j}}.
\end{aligned}$$

Thus it is enough to show that

$$\begin{aligned}
&2^n q^{\sum_{i=1}^n(2i-1)k_i+\sum_{j=1}^n(2j-1)\ell_j} \frac{\prod_{i=1}^n(-q^2;q^2)_{k_i}\prod_{j=1}^n(-q^2;q^2)_{\ell_j}}{\prod_{i,j=1}^n(q^{2k_i}+q^{2\ell_j})}\\
&= q^{|\lambda-\mu|+|\mu-\lambda|+2n(\lambda,\mu)+2n(\mu,\lambda)} \prod_{c\in\lambda}(1+q^{2h_{\lambda\mu}(c)}) \prod_{c\in\mu}(1+q^{2h_{\mu\lambda}(c)}).
\end{aligned}$$

Note that $h_{\lambda\mu}(i,j) = \lambda_i - j + \mu'_j - i + 1 = \lambda_i - j + \sharp\{r : \mu_r \ge j\} - i + 1$. Here $\sharp A$ stands for the cardinality of the set A. For a fixed i, let $n_i = \sharp\{r : \mu_r > \lambda_i\}$ denote the number of parts of μ which is greater than λ_i. Thus we have $\sharp\{r : \mu_r \le \lambda_i\} = n - n_i$. Then, if we draw the Young diagram of λ and fill each cell c with $h_{\lambda\mu}(c)$, then we find that the numbers in the ith row of λ are

$$[n_i - i + 1, k_i] - \{k_i - \ell_r : n_i < r \le n\}.$$

Here we write $[a,b] = \{a, a+1, a+2, \dots, b\}$ for integers $a, b \in \mathbb{Z}$. In fact fix an i, and put $\mu^{(i)} = (\mu^{(i)}_1, \dots, \mu^{(i)}_{n-n_i}) = (\mu_{n_i+1}, \mu_{n_i+2}, \dots, \mu_n)$. Note that the conjugate $\mu^{(i)\prime}$ fits in the box $\lambda_i \times (n - n_i)$. Use [24], I. (1.7) to the partition $\mu^{(i)\prime}$, then we obtain $\{\mu^{(i)\prime}_r + \lambda_i - r : 1 \le r \le \lambda_i\} \uplus \{\lambda_i - 1 + r - \mu^{(i)}_r : 1 \le r \le n - n_i\} = [0, \lambda_i + n - n_i - 1]$. Since $\mu^{(i)\prime}_r = \mu'_r - n_i$, we conclude that $\{h_{\lambda\mu}(i,r) : 1 \le r \le \lambda_i\} \uplus \{k_i - \mu_{r+n_i} : 1 \le r \le n - n_i\} = [n_i - i + 1, k_i]$.

Put $m_j = \sharp\{r : \lambda_r \ge \mu_j\}$ for each $1 \le j \le n$. The same argument shows that, if we write the Young diagram of μ and fill each cell c with $h_{\mu\lambda}(c)$, then the numbers in the jth row of μ are

$$[m_j - j + 1, \ell_j] - \{\ell_j - k_r : m_j < r \le n\}.$$

Thus it is enough to show that

$$\begin{aligned}
&2^n q^{\sum_{i=1}^n(2i-1)k_i+\sum_{j=1}^n(2j-1)\ell_j} \frac{\prod_{i=1}^n(-q^2;q^2)_{k_i}\prod_{j=1}^n(-q^2;q^2)_{\ell_j}}{\prod_{i,j=1}^n(q^{2k_i}+q^{2\ell_j})}\\
&= q^{|\lambda-\mu|+|\mu-\lambda|+2n(\lambda,\mu)+2n(\mu,\lambda)} \frac{\prod_{i=1}^n\prod_{r=n_i-i+1}^{k_i}\left(1+q^{2r}\right)\prod_{j=1}^n\prod_{r=m_j-j+1}^{\ell_j}\left(1+q^{2r}\right)}{\prod_{\lambda_i\ge\mu_j}\left(1+q^{2(k_i-\ell_j)}\right)\prod_{\lambda_i<\mu_j}\left(1+q^{2(\ell_i-k_j)}\right)}
\end{aligned}$$

First it is easy to see that

$$\prod_{\mu_j\le\lambda_i}(1+q^{2(k_i-\ell_j)})\prod_{\lambda_i<\mu_j}(1+q^{2(\ell_i-k_j)})=q^{-2P(\lambda,\mu)}\prod_{i,j=1}^{n}(q^{2k_i}+q^{2\ell_j}),$$

where

$$P(\lambda,\mu)=\sum_{\lambda_i\ge\mu_j}\ell_j+\sum_{\lambda_i<\mu_j}k_i=\sum_{i=1}^{n}n_ik_i+\sum_{j=1}^{n}m_j\ell_j.$$

Next we claim that

$$\prod_{i=1}^{n}\prod_{r=n_i-i+1}^{k_i}(1+q^{2r})\prod_{j=1}^{n}\prod_{r=m_j-j+1}^{\ell_j}(1+q^{2r})=q^{-2Q(\lambda,\mu)}2^n\prod_{i=1}^{n}(-q^2;q^2)_{k_i}\prod_{j=1}^{n}(-q^2;q^2)_{\ell_j},$$

where

$$Q(\lambda,\mu)=\sum_{\substack{i\\ \lambda_i\ge\mu_i}}\binom{i-1-n_i}{2}+\sum_{\substack{i\\ \lambda_i<\mu_i}}\binom{i-1-m_i}{2}.$$

In fact, let A and B be the sets of lattice points defined by

$$A=\bigcup_{i=1}^{n}\{(i-1,y):n_i\le y\le\lambda_i+n-1\},$$
$$B=\bigcup_{j=1}^{n}\{(x,j-1):m_j\le x\le\mu_j+n-1\}.$$

Then we have $\prod_{i=1}^{n}\prod_{r=n_i-i+1}^{k_i}\left(1+q^{2r}\right)=\prod_{(x,y)\in A}(1+q^{2(y-x)})$ and $\prod_{j=1}^{n}\prod_{r=m_j-j+1}^{\ell_j}\left(1+q^{2r}\right)=\prod_{(x,y)\in B}(1+q^{2(x-y)})$. For example, if $n=4$, $\lambda=(4,3,1,1)$ and $\mu=(3,3)$, then the big circles in Figure 5 are in A and the small circles are in B. The numbers assigned to big circles are $y-x$ and the numbers assigned to small circles are $x-y$. Put $A_1=\bigcup_{i=1}^{n}\{(i-1,y):n_i\le y\le n-1\}$, $B_1=\bigcup_{j=1}^{n}\{(x,j-1):m_i\le x\le n-1\}$, $A_2=\bigcup_{i=1}^{n}\{(i-1,y):n\le y\le\lambda_i+n-1\}$ and $B_2=\bigcup_{j=1}^{n}\{(x,j-1):n\le x\le\mu_j+n-1\}$. Then we have $A=A_1\cup A_2$ and $B=B_1\cup B_2$. It is also easy to see that $A_1\cup B_1=[0,n-1]\times[0,n-1]$, which implies that, as a multi-set,

$$\bigcup_{i=1}^{n}\{|y-i+1|:n_i\le y\le n-1\}\cup\bigcup_{j=1}^{n}\{|x-j+1|:m_i\le x\le n-1\}$$

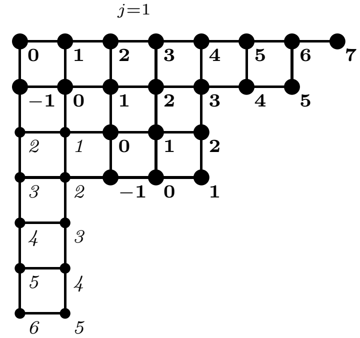

Figure 5: Lattice points

is equal to $\{|x-y|; (x,y) \in [0,n-1]\times[0,n-1]\}$, and is composed of n 0's, $2(n-1)$ 1's, $2(n-2)$ 2's, …, 2 $(n-1)$'s. This shows that

$$\begin{aligned}&\prod_{(x,y)\in A}(1+q^{2(y-x)})\prod_{(x,y)\in B}(1+q^{2(x-y)})\\&=q^{-2Q(\lambda,\mu)}2^n\prod_{i=1}^{n}(-q^2;q^2)_{k_i}\prod_{j=1}^{n}(-q^2;q^2)_{\ell_j}.\end{aligned}$$

In another word we can restate

$$Q(\lambda,\mu)=\sum_{\substack{(x,y)\in A\\x<y}}(y-x)+\sum_{\substack{(x,y)\in B\\x>y}}(x-y).$$

Thus the proof will be done if we prove the following identity:

$$\begin{aligned}&|\lambda-\mu|+|\mu-\lambda|+2n(\lambda,\mu)+2n(\mu,\lambda)+2P(\lambda,\mu)-2Q(\lambda,\mu)\\&=\sum_{i=1}^{n}(2i-1)k_i+\sum_{j=1}^{n}(2j-1)\ell_j.\end{aligned}$$

In the above example, we have $|\lambda-\mu|=3$, $|\mu-\lambda|=n(\mu,\lambda)=0$, $n(\lambda,\mu)=1$, $P(\lambda,\mu)=64$, $Q(\lambda,\mu)=4$, and $3+2+64-4=65=\sum_{i=1}^{4}(2i-1)k_i+\sum_{j=1}^{4}(2j-1)\ell_j$. In the following lemma we prove this identity. □

Lemma 6.5 *Let n be a nonnegative integer, and let λ and μ be partitions which satisfies $\ell(\lambda),\ell(\mu)\le n$. Let $P(\lambda,\mu)$ and $Q(\lambda,\mu)$ be as above. Then the identity*

$$\begin{aligned}&|\lambda-\mu|+|\mu-\lambda|+2n(\lambda,\mu)+2n(\mu,\lambda)+2P(\lambda,\mu)-2Q(\lambda,\mu)\\&=\sum_{i=1}^{n}(2i-1)k_i+\sum_{j=1}^{n}(2j-1)\ell_j.\end{aligned}\tag{58}$$

holds.

Proof. We proceed by induction on n. When $n=1$, assume $\lambda_1\ge\mu_1$. Then it is easy to see that $n(\lambda,\mu)=n(\mu,\lambda)=Q(\lambda,\mu)=0$ and $P(\lambda,\mu)=\mu$. This shows that the left-hand side equals $\lambda_1+\mu_1$ and it coincides with the right-hand sides. In the case $\lambda_1<\mu_1$, we can prove it similarly. Assume $n\ge2$ and (58) holds up to $(n-1)$. Given partitions $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_n)$, we put $\widetilde{\lambda}=(\widetilde{\lambda}_1,\dots,\widetilde{\lambda}_{n-1})=(\lambda_1,\dots,\lambda_{n-1})$ and $\widetilde{\mu}=(\widetilde{\mu}_1,\dots,\widetilde{\mu}_{n-1})=(\mu_1,\dots,\mu_{n-1})$. Further we set $\widetilde{k}_i=\widetilde{\lambda}_i+n-1-i$ and $\widetilde{\ell}_i=\widetilde{\mu}_i+n-1-i$ for $1\le i\le n-1$. Then, by the induction hypothesis, we may assume that (58) holds for $(n-1)$, $\widetilde{\lambda}$, $\widetilde{\mu}$, $\widetilde{k}$ and $\widetilde{\ell}$. First we assume that $\lambda_n\ge\mu_n$. Thus we have $|\lambda-\mu|+|\mu-\lambda|=|\widetilde{\lambda}-\widetilde{\mu}|+|\widetilde{\mu}-\widetilde{\lambda}|+\lambda_n-\mu_n$. From the condition $\lambda_n\ge\mu_n$, we can find an integer s such that $0\le s<n$ and $\mu_s>\lambda_n\ge\mu_{s+1}$ holds. Here we use the convention that $\lambda_0=\mu_0=\infty$. Using this s, we can express the statistics on λ and μ with the statistics on $\widetilde{\lambda}$ and $\widetilde{\mu}$. For example, if we write the Young diagram of λ and μ and fill the cell $(i,j)\in\lambda-\mu$ with the number μ'_j-i-1, then we easily see that

$$\begin{aligned}n(\lambda,\mu)&=n(\widetilde{\lambda},\widetilde{\mu})+(n-s-1)(\lambda_n-\mu_{s+1})+\sum_{i=s+1}^{n-1}(n-i-1)(\mu_i-\mu_{i+1}),\\&=n(\widetilde{\lambda},\widetilde{\mu})+(n-s-1)\lambda_n-\sum_{i=s+1}^{n-1}\mu_i\end{aligned}$$

$$n(\mu,\lambda)=n(\widetilde{\mu},\widetilde{\lambda}).$$

For a fixed i such that $1 \le i < n$, from the fact that $\mu_s > \lambda_n \ge \mu_{s+1}$, it is easy to see that

$$\sharp\{r : \mu_r > \lambda_i\} = \sharp\{r : \widetilde{\mu}_r > \widetilde{\lambda}_i\},$$
$$\sharp\{r : \lambda_r \ge \mu_i\} = \begin{cases} \sharp\{r : \widetilde{\lambda}_r \ge \widetilde{\mu}_i\} & \text{if } 1 \le i \le s, \\ \sharp\{r : \widetilde{\lambda}_r \ge \widetilde{\mu}_i\} + 1 & \text{if } s+1 \le i < n. \end{cases}$$

From these facts we have

$$P(\lambda, \mu) = P(\widetilde{\lambda}, \widetilde{\mu}) + (n-1)^2 + \sum_{j=s+1}^{n-1} \widetilde{\ell}_j + s\lambda_n + n\mu_n,$$
$$Q(\lambda, \mu) = Q(\widetilde{\lambda}, \widetilde{\mu}) + \binom{n-1-s}{2}.$$

Here we used the fact $\sum_{i=1}^{n-1} \sharp\{r : \mu_r > \lambda_i\} + \sum_{j=1}^{n-1} \sharp\{r : \lambda_r \ge \mu_j\} = (n-1)^2$, which is easy to confirm. From these identities, we obtain

$$\begin{aligned} &|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu) \\ &= |\widetilde{\lambda} - \widetilde{\mu}| + |\widetilde{\mu} - \widetilde{\lambda}| + 2n(\widetilde{\lambda}, \widetilde{\mu}) + 2n(\widetilde{\mu}, \widetilde{\lambda}) + 2P(\widetilde{\lambda}, \widetilde{\mu}) - 2Q(\widetilde{\lambda}, \widetilde{\mu}) \\ &+ (2n-1)\lambda_n + (2n-1)\mu_n + 2(n-1)^2. \end{aligned}$$

By the induction hypothesis we have $|\widetilde{\lambda} - \widetilde{\mu}| + |\widetilde{\mu} - \widetilde{\lambda}| + 2n(\widetilde{\lambda}, \widetilde{\mu}) + 2n(\widetilde{\mu}, \widetilde{\lambda}) + 2P(\widetilde{\lambda}, \widetilde{\mu}) - 2Q(\widetilde{\lambda}, \widetilde{\mu}) = \sum_{i=1}^{n-1} (2i-1)\widetilde{k}_i + \sum_{i=1}^{n-1} (2i-1)\widetilde{\ell}_i = \sum_{i=1}^{n-1} (2i-1)k_i + \sum_{i=1}^{n-1} (2i-1)\ell_i - 2(n-1)^2$, and this proves the desired identity. In the case of $\lambda_n < \mu_n$, we may find an integer s which satisfies $0 \le s < n$ and $\lambda_s \ge \mu_n > \lambda_{s+1}$. A similar argument will lead to the desired identity again. □

Proof of Theorem 6.1. We may assume that the number of variables are finite, i.e., $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Assume $N \ge n$ is a positive integer. Let T and S be two n by N rectangular matrices defined by

$$T = \left(x_i^{N-j}\right)_{j=1,\dots,N}^{i=1,\dots,n}, \qquad S = \left(y_i^{N-j}\right)_{j=1,\dots,N}^{i=1,\dots,n}.$$

Let A be an N by N square matrix defined by

$$A = \left(\frac{(-q^2; q^2)_{N-i}}{(q^2; q^2)_{N-i}} \frac{(-q^2; q^2)_{N-j}}{(q^2; q^2)_{N-j}} \frac{2}{q^{i-j} + q^{j-i}} \right)_{i,j=1,\dots,N}.$$

Now we compute $\lim_{N\to\infty} \det {}^tTAS$ in two different ways. By the Cauchy-Binet formula (21), we have

$$\det {}^tTAS = \sum_{\substack{I \subseteq [N] \\ \sharp I = n}} \sum_{\substack{J \subseteq [N] \\ \sharp J = n}} \det T_I \det A_J^I \det S_J.$$

Put $I = \{i_1, \dots, i_n\}$ and $J = \{j_1, \dots, j_n\}$. Then there exist partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ such that $\lambda_1, \mu_1 \le N - n$ and we can write $N - i_r = \lambda_r + n - r$ and $N - j_r = \mu_r + n - r$ for $r = 1, \dots, n$. Then it is easy to see that $\det T_I = \det\left(x_i^{\lambda_j + n - j}\right) = \Delta(x) s_\lambda(x)$ and

$\det S_J = \det\left(y_i^{\mu_j+n-j}\right) = \Delta(y)s_\mu(y)$. As before we put $k_r = \lambda_r + n - r$ and $\ell_r = \mu_r + n - r$ for $r = 1, \dots, n$. Then, by Lemma 6.4, we obtain

$$\det A_J^I = \det\left[\frac{(-q^2;q^2)_{k_i}}{(q^2;q^2)_{k_i}}\frac{(-q^2;q^2)_{\ell_j}}{(q^2;q^2)_{\ell_j}}\frac{2}{q^{k_i-\ell_j}+q^{\ell_j-k_i}}\right] = q^{|\lambda-\mu|+|\mu-\lambda|}J_{\lambda\mu}(q^2).$$

On the other hand, by Lemma 6.2, we have

$$\lim_{N\to\infty}\det {}^tTAS = \det\left[\frac{(-qx_i;q^2)_\infty}{(qx_i;q^2)_\infty}\frac{(-qy_j;q^2)_\infty}{(qy_j;q^2)_\infty}\frac{1}{1-x_iy_j}\right]_{i,j=1,\dots,n}.$$

Thus (53) is an immediate consequence of (56). This proves the theorem.
□

7 Appendix: A variant of the Sundquist formula

We give here some variant (both a statement and a proof) of the Sundquist formula [32]. Indeed, we establish the following Theorem 7.1. Although the initial proof of the theorem was made by employing the basic identity in §2, it did not use directly the minor summation formula and was also, in fact, complicated. Thus we decided to treat this in the Appendix. The proof presented here is the one followed by the suggestion of the referee.

Theorem 7.1 *It holds that*

$$\begin{aligned}
&\mathrm{Pf}\left(\frac{y_i-y_j}{a+b(x_i+x_j)+cx_ix_j}\right)_{1\le i,j\le 2n}\times\prod_{1\le i<j\le 2n}\{a+b(x_i+x_j)+cx_ix_j\}\\
&=(ac-b^2)^{\binom{n}{2}}\sum_{\substack{I\subseteq[2n]\\ \sharp I=n}}(-1)^{|I|-\binom{n+1}{2}}y_I\Delta_I(x)\Delta_{\overline{I}}(x)J_I(x)J_{\overline{I}}(x)\\
&=(ac-b^2)^{\binom{n}{2}}\sum_{\substack{I\subseteq[2n]\\ \sharp I=n\\ i_1<j_1}}(-1)^{|I|-\binom{n+1}{2}}(y_I+(-1)^ny_{\overline{I}})\Delta_I(x)\Delta_{\overline{I}}(x)J_I(x)J_{\overline{I}}(x),
\end{aligned}\tag{59}$$

where the sum runs over all n-elements subset $I=\{i_1<\cdots<i_n\}$ of $[2n]=\{1,2,\dots,2n\}$ such that $i_1<j_1$ and $|I|=i_1+\cdots+i_n$. Moreover $\overline{I}=\{j_1<\cdots<j_n\}$ is the complementary subset of I in $[2n]$ and

$$\Delta_I(x)=\prod_{\substack{i,j\in I\\ i<j}}(x_i-x_j),$$

$$J_I(x)=J_I(x;a,b,c)=\prod_{\substack{i,j\in I\\ i<j}}\{a+b(x_i+x_j)+cx_ix_j\},$$

$$y_I=\prod_{i\in I}y_i.$$

In particular, if the relation $ac=b^2$ holds then

$$\mathrm{Pf}\left(\frac{y_i-y_j}{a+b(x_i+x_j)+cx_ix_j}\right)_{1\le i,j\le 2n}=0.$$

Example 7.2 *In the case of $n = 2$, if we put $a = c = 1$ and $b = 0$ then the theorem above reads*

$$\begin{aligned}
&\mathrm{Pf}\left[\frac{y_i - y_j}{1 + x_i x_j}\right]_{1\le i<j\le 4} \times \prod_{1\le i<j\le 4}(1 + x_i x_j)\\
&= (y_1y_2 + y_3y_4)(x_1 - x_2)(x_3 - x_4)(1 + x_1x_2)(1 + x_3x_4)\\
&- (y_1y_3 + y_2y_4)(x_1 - x_3)(x_2 - x_4)(1 + x_1x_3)(1 + x_2x_4)\\
&+ (y_1y_4 + y_2y_3)(x_1 - x_4)(x_2 - x_3)(1 + x_1x_4)(1 + x_2x_3)
\end{aligned}$$

Proof of Theorem 7.1. Since

$$a + b(x_i + x_j) + cx_ix_j = (\sqrt{c}x_i + \frac{b}{\sqrt{c}})(\sqrt{c}x_j + \frac{b}{\sqrt{c}}) + a - \frac{b^2}{c}$$

it is enough to show the theorem for the case $a = c = 1$ and $b = 0$.

Moreover, since the second equality follows immediately from the first one if one notes the fact that $|I| + |\overline{I}| \equiv n \mod 2$, we give a proof of the first equality. First, we notice the following

Lemma 7.3 *The coefficient of y_I in the Pfaffian $\mathrm{Pf}\left(\frac{y_j - y_i}{1+x_ix_j}\right)$ is equal to the Pfaffian of the following skew symmetric matrix T_I: The (i, j) entry $(T_I)_{ij}$ of T_I is given by*

$$(T_I)_{ij} = \begin{cases} 1/(1 + x_ix_j) & \text{if} \quad i \in I \text{ and } j \in \overline{I}, \\ -1/(1 + x_ix_j) & \text{if} \quad i \notin \overline{I} \text{ and } j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall the definition of a Pfaffian:

$$\mathrm{Pf}\left(\frac{y_j - y_i}{1 + x_ix_j}\right) = \sum_{\sigma} \epsilon(\sigma)\, \frac{y_{\sigma_1} - y_{\sigma_2}}{1 + x_{\sigma_1}x_{\sigma_2}} \cdots \frac{y_{\sigma_{2n-1}} - y_{\sigma_{2n}}}{1 + x_{\sigma_{2n-1}}x_{\sigma_{2n}}},$$

where the summation is over all partitions $\sigma = \{\{\sigma_1, \sigma_2\}_<, \ldots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-elements blocks, and $\epsilon(\sigma) = \epsilon(\sigma_1, \ldots, \sigma_{2n})$ denotes the sign of $\sigma \in \mathfrak{S}_{2n}$. From this expression we immediately see that the coefficient of y_I is given by

$$\sum_{\sigma} \epsilon(\sigma) \prod_{\substack{\sigma_{2k-1}\in I\\ \sigma_{2k}\in \overline{I}}} \frac{1}{1 + x_{\sigma_{2k-1}}x_{\sigma_{2k}}} \prod_{\substack{\sigma_{2k-1}\in \overline{I}\\ \sigma_{2k}\in I}} \frac{-1}{1 + x_{\sigma_{2k-1}}x_{\sigma_{2k}}}.$$

Note here that when σ is subject to either the conditions $\sigma_{2k-1} \in I, \sigma_{2k} \in I$ or $\sigma_{2k-1} \in \overline{I}, \sigma_{2k} \in \overline{I}$, the corresponding term disappears in the sum. Hence the assertion follows easily. □

By this lemma, in order to prove the theorem (for $a = c = 1$, $b = 0$), it suffices to show that

$$\mathrm{Pf}(T_I) = (-1)^{|I|-\binom{n+1}{2}}\Delta_I(x)\Delta_{\overline{I}}(x) \prod_{i\in I, j\in \overline{I}} \frac{1}{1 + x_ix_j}. \tag{60}$$

This Pfaffian $\mathrm{Pf}(T_I)$ is computed by using a relation between Pfaffians of special type and determinants, and the Cauchy determinant formula as follows: Recalling $I = \{i_1 < i_2 < \ldots < i_n\} \subseteq [2n]$ and $\overline{I} = \{j_1 < j_2 < \ldots < j_n\} \subseteq [2n]$, we first notice that

$$\mathrm{Pf}(T_I) = (-1)^{|I|-\binom{n+1}{2}}\, \mathrm{Pf}\begin{pmatrix} 0 & X_I \\ -X_I & 0 \end{pmatrix},$$

where the $n \times n$ matrix X_I is determined by $(X_I)_{k\ell} = \frac{1}{1+x_{i_k}x_{j_\ell}}$ ($i_k \in I$, $j_\ell \in \overline{I}$) because the number of the column-row changes for obtaining $\mathrm{Pf}\begin{pmatrix} 0 & X_I \\ -X_I & 0 \end{pmatrix}$ from $\mathrm{Pf}(T_I)$ equals $(i_1 - 1) + (i_2 - 2) + \cdots + (i_n - n) = |I| - \frac{1}{2}n(n+1) = |I| - \binom{n+1}{2}$. Moreover, since

$$\mathrm{Pf}\begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} = (-1)^{\binom{n}{2}} \det(X)$$

for any $n \times n$ matrix X, the Cauchy determinant formula claims that

$$\begin{aligned}\mathrm{Pf}\begin{pmatrix} 0 & X_I \\ -X_I & 0 \end{pmatrix} &= (-1)^{\binom{n}{2}} \Delta_I(-x)\Delta_{\overline{I}}(x) \prod_{i\in I, j\in\overline{I}} \frac{1}{1-(-x_i)x_j} \\ &= \Delta_I(x)\Delta_{\overline{I}}(x) \prod_{i\in I, j\in\overline{I}} \frac{1}{1+x_ix_j},\end{aligned}$$

whence the equation (60) follows. Multiplying the factor $\prod_{1\le i,j\le 2n}(1 + x_ix_j)$ to the both sides of (60), we obtain the desired identity. This proves the theorem. $\Box$

As a corollary of this theorem we obtain the Sundquist identity [32]. The Sundquist identity is a two-variable generalization of $\mathrm{Pf}(\frac{x_j-x_i}{1-tx_ix_j})$ and it is considered as a Pfaffian version of Cauchy determinant formula, whose evaluation is given by [31] (see also Lemma 8 in [8]):

$$\mathrm{Pf}(\frac{x_j - x_i}{1 - tx_ix_j})_{1\le i<j\le 2n} = t^{n(n-1)} \frac{\prod_{1\le i<j\le 2n}(x_j - x_i)}{\prod_{1\le i<j\le 2n}(1 - tx_ix_j)}.$$

Corollary 7.4 *(Sundquist)*

$$\mathrm{Pf}\left(\frac{y_i - y_j}{1 + x_ix_j}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} (1 + x_ix_j) = \sum_{\lambda,\mu} a_{\lambda+\delta_n,\mu+\delta_n}(x, y),$$

where the sums runs over pairs of partitions

$$\lambda = (\alpha_1, \cdots, \alpha_p|\alpha_1 + 1, \cdots, \alpha_p + 1), \mu = (\beta_1, \cdots, \beta_p|\beta_1 + 1, \cdots, \beta_p + 1)$$

in Frobenius notation with $\alpha_1, \beta_1 < n - 1$. *Also, for* α *and* β *partitions (compositions, in general) of length* n, *we put*

$$a_{\alpha,\beta}(x, y) = \sum_{\sigma\in\mathfrak{S}_{2n}} \epsilon(\sigma)\sigma(x_1^{\alpha_1}y_1 \cdots x_n^{\alpha_n}y_nx_{n+1}^{\beta_1} \cdots x_{2n}^{\beta_n}),$$

where $\sigma \in \mathfrak{S}_{2n}$ *acts on each of two sets of variables* $\{x_1, \cdots, x_n\}$ *and* $\{y_1, \cdots, y_n\}$ *by permuting indices, and* $\delta_n = (n - 1, n - 2, \cdots, 0)$. $\Box$

We have already given in [12] a way of reduction of this corollary from Theorem 7.1 by using the expansion

$$\prod_{1\le i<j\le n} (1 + x_ix_j) = \sum_{\lambda=(\alpha_1,\cdots,\alpha_p|\alpha_1+1,\cdots,\alpha_p+1)} s_\lambda(x_1, \cdots, x_n),$$

where $s_\lambda = s_\lambda(x_1, \cdots, x_n)$ are the Schur functions, so we omit the proof.

Suppose n is even. If we put $y_i = 1$ for all $1 \le i \le 2n$ in Theorem 7.1, it is immediate to see the

Corollary 7.5

$$\sum_{\substack{I\subseteq[2n]\\ \sharp I=n\\ i_1<j_1}}(-1)^{|I|}\Delta_I(x)\Delta_I(x)J_I(x)J_{\overline{I}}(x)=0$$

holds for even n. □

Example 7.6 *When $n=2$ we have*

$$\begin{aligned}&(x_1-x_2)(x_3-x_4)(1+x_1x_2)(1+x_3x_4)\\ -&(x_1-x_3)(x_2-x_4)(1+x_1x_3)(1+x_2x_4)\\ +&(x_1-x_4)(x_2-x_3)(1+x_1x_4)(1+x_2x_3)=0.\end{aligned}$$

Remark 7.7 *It is naturally thought the formula as the identity of two variables relevant to a A_n-type root system. It would be interesting to establish the B_n, C_n, D_n-analogues of Theorem 7.1 like in [7] for the generalization of the Littlewood formulas to the classical groups.*

Acknowledgements. The authors would like to thank the referee for his/her many valuable comments and suggestions.

Note Added in Proof In a private communication ("Kawanaka's q-Cauchy identity and rational universal character"), S. Okada has suggested that (53) can be proved by the rational universal characters of the classical groups (see K. Koike's paper: "On the decomposition of tensor products of the representations of the classical groups: By means of the universal characters", Adv. Math. **74** (1989), 57–86.).

References

[1] D.M. Bressoud, *Proofs and Confirmations*, Cambridge U.P.

[2] D.M. Bressoud, *Identities for Schur functions and plane partitions*, Ramanujan J. **4** (2000) 69-80.

[3] A. Dress and W. Wenzel, *A simple proof of an identity concerning pfaffians of skew symmetric matrices*, Adv. Math. **112** (1995), 120–134.

[4] A. Gyoja, K. Nishiyama and K. Taniguchi, *Kawanaka invariants for representations of Weyl groups*, J. Alg., 225 (2000), 842 - 871.

[5] I. Gessel and G. Viennot, *Determinants, Paths, and Plane Partitions*, preprint (1989).

[6] R. Hirota, *Mathematical aspect of the soliton theory from a direct methods point of view*, Iwanami Shoten (1992), in Japanese.

[7] M. Ishikawa, S. Okada and M. Wakayama, *Applications of minor summation formulas I, Littlewood's formulas*, J. Alg. **183** (1996), 193–216.

[8] M. Ishikawa and M. Wakayama, *Minor summation formula of Pfaffians*, Linear and Multilinear Alg. **39** (1995), 285-305.

[9] M. Ishikawa and M. Wakayama, *Minor summation formula of Pfaffians and Schur functions identities*, Proc.Japan Acad., Ser.A **71** (1995), 54–57.

[10] M. Ishikawa and M. Wakayama, *New Schur function series*, J. Alg, **208** (1998), 480–525.

[11] M. Ishikawa and M. Wakayama, *Applications of minor summation formulas II, Pfaffians of Schur polynomials*, J. Combin. Th. Ser. A **88** (1999), 136–157.

[12] M. Ishikawa and M. Wakayama, *Minor summation formula of Pfaffians, Survey and a new identity*, Adv. Stud. Pure Math. **28** (2000), 133–142.

[13] F. Jouhet and J. Zeng, *Some New Identities for Schur Functions*, Adv. in Applied Math. **27** (2001), 493–509.

[14] F. Jouhet and J. Zeng, *New Identities of Hall-Littlewood Polynomials and Applications*, to appear in Ramanujan J.

[15] N. Kawanaka, *A q-series identity involving Schur functions and related topics*, Osaka J. Math. **36** (1999), 157–176.

[16] N. Kawanaka, *A q-Cauchy identity involving Schur functions and imprimitive complex reflection groups*, Osaka J. Math. **38** (2001), 775–810.

[17] K. Kinoshita and M. Wakayama, *Explicit Capelli identities for skew symmetric matrices*, Proc. Edinburgh Math. Soc. **45** (2002), 449–465.

[18] D. Knuth, *Overlapping pfaffians*, Electronic J. of Combi. **3** (1996), 151–163.

[19] C. Krattenthaler, *Determinant identities and a generalization of the number of totally symmetric self-complementary plane partitions*, Electron. J. Combin. **4** (1) (1997), ♯R27, 62.

[20] C. Krattenthaler, *Advanced Determinant Calculus*, Seminaire Lotharingien Combin. **42** (”The Andrews Festschrift”) (1999), Article B42q, 67.

[21] A. Lascoux, *Littlewood's formulas for characters of orthogonal and symplectic groups*, Algebraics Combinatorics and Quantum Groups (2003).

[22] J.-G. Luque and J.-Y. Thibon, *Hankel hyperdeterminants and Selberg integrals* J. Phys. A: Math. Gen. 36 (16 May 2003) 5267-5292.

[23] J.-G. Luque and J.-Y. Thibon, *Pfaffian and hafnian identities in shuffle algebras*, Adv. Appl. Math. **29** (2002), 620–646.

[24] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd Edition*, Oxford University Press, (1995).

[25] W.H. Mills, D.P. Robbins and H. Rumsey, Jr., *Self-Complementary Totally Symmetric Plane Partitions*, J. Combin. Th. Ser. A **42** (1986), 277–292.

[26] S. Okada, *On the generating functions for certain classes of plane partitions*, J. Combin. Th. Ser. A **51** (1989), 1–23.

[27] S. Okada, *Applications of minor-summation formulas to rectangular-shaped representations of classical groups*, J. Alg. **205** (1998), 337–367.

[28] D.P. Robbins and H. Rumsey Jr., *Determinants and Alternating Sign Matrices*, Adv. Math. **62** (1986), 169–184.

[29] I. Schur, Aufgabe 569, Arch. Math. Phys. (3) **27** (1918), 163.

[30] R. P. Stanley, *Enumerative Combinatorics, Volume I, II*, Cambridge University Press, (1997), (1999).

[31] J. Stembridge, *Nonintersecting paths, pfaffians and plane partitions*, Adv. Math. **83** (1990), 96–131.

[32] T. Sundquist *Two variable Pfaffian identities and symmetric functions* J. Alg. Combin. **5** (1996), 135–148.

[33] C.A. Tracy and H. Widom, *A Limit Theorem for Shifted Schur Measures*, Duke Math. J. **123** (2004) 178–208.

Masao ISHIKAWA, Department of Mathematics, Faculty of Education, Tottori University, Tottori 680-8551, Japan

E-mail address: ishikawa@fed.tottori-u.ac.jp

Masato Wakayama, Faculty of Mathematics, Kyushu Unversity, Hakozaki, Fukuoka 812-8581, Japan

E-mail address: wakayama@math.kyushu-u.ac.jp